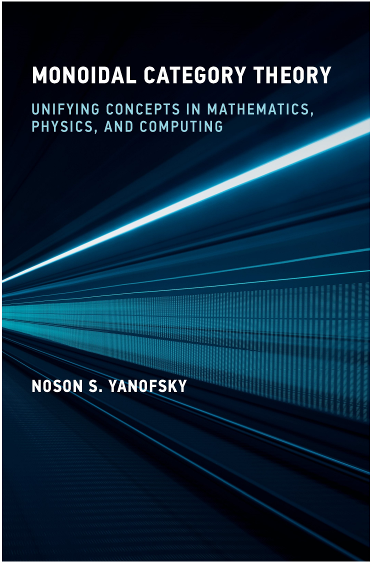
Monoidal Category Theory: Unifying concepts in Mathematics, Physics, and Computing



Chapter 5:

Monoidal Categories

- Chapter 5: Monoidal Categories
  - Section 5.1: Strict Monoidal Categories
  - Section 5.2: Cartesian Categories
  - Section 5.3: Monoidal Categories
  - Section 5.4: Coherence Theory
  - Section 5.5: String Diagrams
  - Section 5.6: Mini-course: Advanced Linear Algebra

Introduction

- Many times in these pages we have seen that objects and morphisms in certain categories can be combined.
- The first time we met this idea was with sets. We saw the formation of the Cartesian product in the category $\mathbb{S}\mathrm{et}$. Given two sets S and T, we form $S \times T$.
- Furthermore, given two set maps $f\colon S \longrightarrow T$ and $g\colon S' \longrightarrow T'$, we form the map $f \times g\colon S \times S' \longrightarrow T \times T'$.
- We also saw that we can combine sets by taking their disjoint union.
- Other examples of combining include taking the product of two partial orders and two groups.
- All these categories where one can combine objects and morphisms are examples of monoidal categories.
- The word "monoidal" comes from the fact that these categories with extra structure are reminiscent of monoids where elements are combined.

Introduction

- Monoidal categories come in many different varieties. The differences depend on what rules the combination of objects and morphisms follow.
- To get a feel for this, let us remember some basic arithmetic.
- There are operations $\otimes$ on numbers (such as $+$ or $\times$) that satisfy the associativity axiom, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.
- There are also operations (such as – and $\div$) that fail this associativity axiom.
- There are similar notions about commutativity where there are some operations $\otimes$ (such as $+$ or $\times$) that satisfy the commutativity axiom and there are some operations (such as – and $\div$) that do not..
- The story with categories is even more varied. We will see that for categories there can be many different possible relationships between $a \otimes (b \otimes c)$ and $(a \otimes b) \otimes c$. Similarly, for commutativity. The many possibilities make the theory of monoidal categories very rich.

- Chapter 5: Monoidal Categories
  - Section 5.1 Strict Monoidal Categories
    - Definitions
    - Examples
    - Examples With Matrices

Basic Definitions

We begin with the simplest possible type of a monoidal category.

Definition

A **strict monoidal category** $(\mathbb{A}, \otimes, I)$ *is a category* $\mathbb{A}$ *with the following extra structure:*

- *A way of combining objects and morphisms: a bifunctor* $\otimes\colon \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ *called the* **tensor product** *or* **monoidal product**, *and*
- *An object* I *of* $\mathbb{A}$ *called the* **unit**.

This extra structure satisfies the following requirements

- *The bifunctor* $\otimes$ *is associative: for all objects* a, b, *and* c *of* $\mathbb{A}$, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$. *Similarly, for all morphisms* f, g, *and* h, $f \otimes (g \otimes h) = (f \otimes g) \otimes h$. *We say the tensor product is* **strictly associative**.
- *The* I *acts like a two-sided unit of* $\otimes$: *for all objects* a *of* $\mathbb{A}$, $a \otimes I = a = I \otimes a$.

Examples

Example

- $(\mathbf{N}, +, 0)$.
- *A simple example of a strict monoidal category is the natural numbers with addition.*
- *The objects of the category are the natural numbers.*
- *The morphisms are only identities.*
- *The tensor product is addition and the unit is* 0.
- *It is also easy to see that for the same discrete category* $\mathbf{N}$, *the multiplication and unit* 1 *form a strict monoidal category* $(\mathbf{N}, \cdot, 1)$.

Examples

Example

- *We just saw the discrete category* $(\mathbf{N}, +, 0)$ *is a strict monoidal category.*
- *If we look at* $\mathbf{N}$ *as a total order, then it is also a strict monoidal category.*
- *One must check that the bifunctor preserves addition, i.e., If* $m \leq m'$ *and* $n \leq n'$ *then* $m + n \leq m' + n'$.
- *This not only works for addition but also for multiplication, which means that* $(\mathbf{N}, \cdot, 1)$ *is a strict monoidal category.*
- *These two strict monoidal operations are not independent of each other. Rather, the multiplication distributes over the addition.*
- *Along the same lines, the totally ordered real numbers* $\mathbf{R}$ *has two strict monoidal category structures* $(\mathbf{R}, +, 0)$ *and* $(\mathbf{R}^+, \cdot, 1)$. *These two monoidal structures are related as well.*

Example

- $(\Sigma^*, \bullet, \emptyset)$.
- *An example of a strict monoidal category that will be very important in the coming pages is the monoid of strings in an alphabet.*
- *Let Σ be an alphabet, that is, a finite set of symbols (or letters).*
- *The set of all strings (or words, or lists) of symbols in Σ, denoted Σ^*, forms a monoidal category.*
- *The objects of the category are strings and the only morphisms are identity morphisms.*
- *The tensor product of the monoidal category is concatenation $\bullet$ (combining one string after another) . That is, given two strings, w and w', their tensor is simply their concatenation $w \bullet w'$. This operation is associative.*

Example (Continued.)

- *The empty string* $\emptyset$ *is the unit:* $w \bullet \emptyset = w = \emptyset \bullet w$.
- *As special instances of such strict monoidal categories, consider the strict monoidal categories of strings in one symbol,* $(\{1\}^*, \bullet, \emptyset)$ *and strings of two symbols* $(\{0, 1\}^*, \bullet, \emptyset)$.

The collections of objects in the above examples are all monoids. Let us make a general statement about all monoids.

Example

$(M, \star, e)$. *Any monoid can be thought of as a strict monoidal category. The category is* $d(M)$, *the discrete category of elements of* M. *The tensor product is the monoid multiplication* $\star$, *and the unit of the strict monoidal category is the unit of the monoid.*

Notice that category theorists can think of a monoid as a category in at least two different ways.

- On the one hand, they are one-object categories where the morphisms come and go to the single object.
- On the other hand, a monoid is a discrete category with a monoidal category structure where the tensor product is the monoid multiplication.

We have to specify what we mean.

Example

- $(P, \wedge, 1)$.
- *Any partial order category* $(P, \leq)$ *that has products is a strict monoidal category.*
- *We call the product* **meet** *and write it as* $\wedge$.
- *For any three objects,* $p, q,$ *and* r*, we have* $p \wedge (q \wedge r) = (p \wedge q) \wedge r$*, which means the meet is associative.*
- *The unit is the terminal object,* 1*, which satisfies* $p \wedge 1 = p$.
- *For any objects,* p *and* q*, we have* $p \wedge q = q \wedge p$*, which means that meet is commutative.*
- *Such a partial order is called a* **bounded meet semilattice**.

Example

- *The same is true for a partial order category with coproducts where the operation is called* **join** *and denoted as* $\vee$.
- *The unit is the initial object,* 0.
- *In that case, we have the strict monoidal category* $(P, \vee, 0)$.
- *Such a partial order is called a* **bounded join semilattice**.
- *A special case of partial order category is the following:* $(\mathbf{2} = \{0, 1\}, \wedge, 1)$. *The partial order with two elements such that* $0 \leq 1$ *is a strict monoidal category. For completeness, let us just give the monoidal structure:* $0 \wedge 0 = 0, 0 \wedge 1 = 0, 1 \wedge 0 = 0$ *, and* $1 \wedge 1 = 1$. *From these we can see the associativity and the fact that* 1 *is the unit.*

Examples

The next example will be very important in coherence theory.

Example

- $(\mathbb{A}^{\mathbb{A}}, \circ, Id_{\mathbb{A}})$.
- *Let* $\mathbb{A}$ *be any category.*
- *The category* $\mathbb{A}^{\mathbb{A}}$ *consists of functors from* $\mathbb{A}$ *to* $\mathbb{A}$ *(endofunctors) and natural transformations between such functors.*
- *This category has a strict monoidal category structure.*
- *The tensor product of objects is the composition* $\circ$. *In detail, if* $F \colon \mathbb{A} \longrightarrow \mathbb{A}$ *and* $G \colon \mathbb{A} \longrightarrow \mathbb{A}$, *then the tensor product is* $F \circ G$.
- *Notice that both* $F \circ G$ *and* $G \circ F$ *exist but need not be equal.*

Example (Continued.)

- *The tensor of morphisms (natural transformations)* $\alpha\colon F \Longrightarrow F'$ *and* $\beta\colon G \Longrightarrow G'$ *is* $\beta \circ \alpha = \beta \circ_H \alpha$*, where* $\circ_H$ *is horizontal composition of natural transformations.*
- *Vertical composition of natural transformations is the regular composition of maps in* $\mathbb{A}^{\mathbb{A}}$*.*
- *The unit of the monoidal structure is the identity functor* $Id_{\mathbb{A}}$*.*

Examples With Matrices

Remark

- *Some foreshadowing is warranted.*
- *The category of matrices and the category of finite dimensional vector spaces are intimately related.*
- *In this chapter, we will show that the category of matrices has two monoidal structures which we will denote as* $\oplus$ *and* $\otimes$*.*
- *Related to these are two monidal structures on the category of finite dimensional vector spaces which are denoted by the same symbols.*
- *Exactly how matrices and vector spaces are related and how all these different structure are united, will be formalized in Chapter 6. Keep in mind the larger picture while going through the technical details.*

Examples With Matrices

Example

$(\mathbf{K}\mathbb{M}\mathrm{at}, \oplus, 0)$. *The category* $\mathbf{K}\mathbb{M}\mathrm{at}$ *has a strict monoidal structure called the* **direct sum** *which is denoted by* $\oplus$. *On objects, the monoidal structure is defined as* $m \oplus n = m + n$. *Remember that a morphism* $A\colon n \longrightarrow m$ *corresponds to an* m *by* n *matrix and is denoted as* $A_{m,n}$. *The* m *by* n *matrix with all zeros is denoted as* $0_{m,n}$. *The direct sum is defined as*

$$A_{m,n} \oplus B_{m',n'} = \begin{bmatrix} A_{m,n} & 0_{m,n'} \\ 0_{m',n} & B_{m',n'} \end{bmatrix}$$

which is an $m + m'$ *by* $n + n'$ *matrix.*

Examples With Matrices

Example (Continued.)

Formally, the direct sum of matrices is a function

$$\oplus : \mathbb{C}^{m\times n} \times \mathbb{C}^{m'\times n'} \longrightarrow \mathbb{C}^{(m+m')\times(n+n')}$$

and is defined as

$$(A \oplus B)[j,k] = \begin{cases} A[j,k] & : \textit{if } j \leq m \textit{ and } k \leq n \\ B[j-m, k-n] & : \textit{if } j > m \textit{ and } k > n \\ 0 & : \textit{otherwise.} \end{cases}$$

Example (Continued.)

This operation preserves matrix multiplication (which is morphism composition in this category):

$$
\begin{aligned}
(A_{m,n} \oplus B_{m',n'}) \cdot (A'_{n,p} \oplus B'_{n',p'}) &= \begin{bmatrix} A_{m,n} & 0_{m,n'} \\ 0_{m',n} & B_{m',n'} \end{bmatrix} \cdot \begin{bmatrix} A'_{n,p} & 0_{n,p'} \\ 0_{n',p} & B'_{n',p'} \end{bmatrix} \\
&= \begin{bmatrix} A_{m,n} \cdot A'_{n,p} & 0_{m,p'} \\ 0_{m',p} & B_{m',n'} \cdot B'_{n',p'} \end{bmatrix} \\
&= (A_{m,n} \cdot A'_{n,p}) \oplus (B_{m',n'} \cdot B_{n',p'}).
\end{aligned}
$$

This equation is yet another instance of the ubiquitous interchange law (See Important Categorical Idea).

Example (Continued.)

The unit for $\oplus$ *is* 0*, i.e.,* $n \oplus 0 = n + 0 = n$*. We are employing a morphism which represents a (non-existent) zero-by-zero matrix with nothing in it.*
One can see that $\oplus$ *is strictly associative as follows:*

$$
\begin{aligned}
(A_{m,n} \oplus B_{m',n'}) \oplus C_{m'',n''} &= \begin{bmatrix} \begin{bmatrix} A_{m,n} & 0_{m,n'} \\ 0_{m',n} & B_{m',n'} \end{bmatrix} & 0_{m+m',n''} \\ 0_{m'',n+n'} & C_{m'',n''} \end{bmatrix} \\
&= \begin{bmatrix} A_{m,n} & 0_{m,n'} & 0_{m,n''} \\ 0_{m',n} & B_{m',n'} & 0_{m',n''} \\ 0_{m'',n} & 0_{m'',n'} & C_{m'',n''} \end{bmatrix} \\
&= \begin{bmatrix} A_{m,n} & 0_{m,n'+n''} \\ 0_{m'+m'',n} & \begin{bmatrix} B_{m',n'} & 0_{m',n''} \\ 0_{m'',n'} & C_{m'',n''} \end{bmatrix} \end{bmatrix} \\
&= A_{m,n} \oplus (B_{m',n'} \oplus C_{m'',n''}).
\end{aligned}
$$

Examples With Matrices

There is another strict monoidal structure on $\mathbf{K}\mathbb{M}\mathrm{at}$.

Example

$(\mathbf{K}\mathbb{M}\mathrm{at}, \otimes, 1)$. *On objects,* $\otimes$ *is defined as* $m \otimes n = m \cdot n$. *The tensor of* $A_{m,n}$ *with* $B_{m',n'}$ *is the* **Kronecker product of matrices** $A_{m,n} \otimes B_{m',n'}$*, which is defined as follows: every entry of* $A_{m,n}$ *is scalar multiplied with the matrix* $B_{m',n'}$*. Examples are as follows.*

Examples With Matrices

Example (Continued.)

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_1 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_2 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_3 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_0 b_2 \\ a_1 b_0 \\ a_1 b_1 \\ a_1 b_2 \\ a_2 b_0 \\ a_2 b_1 \\ a_2 b_2 \\ a_3 b_0 \\ a_3 b_1 \\ a_3 b_2 \end{bmatrix}.$$

The Kronecker product of two vectors. Every entry in the first vector is scalar multiplied with the second vector.

Examples With Matrices

Example (Continued.)

For matrices

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix},$$

the Kronecker product is

$$A \otimes B = \begin{bmatrix} a_{0,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{0,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{1,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{0,0} \cdot b_{0,0} & a_{0,0} \cdot b_{0,1} & a_{0,0} \cdot b_{0,2} & a_{0,1} \cdot b_{0,0} & a_{0,1} \cdot b_{0,1} & a_{0,1} \cdot b_{0,2} \\ a_{0,0} \cdot b_{1,0} & a_{0,0} \cdot b_{1,1} & a_{0,0} \cdot b_{1,2} & a_{0,1} \cdot b_{1,0} & a_{0,1} \cdot b_{1,1} & a_{0,1} \cdot b_{1,2} \\ a_{0,0} \cdot b_{2,0} & a_{0,0} \cdot b_{2,1} & a_{0,0} \cdot b_{2,2} & a_{0,1} \cdot b_{2,0} & a_{0,1} \cdot b_{2,1} & a_{0,1} \cdot b_{2,2} \\ a_{1,0} \cdot b_{0,0} & a_{1,0} \cdot b_{0,1} & a_{1,0} \cdot b_{0,2} & a_{1,1} \cdot b_{0,0} & a_{1,1} \cdot b_{0,1} & a_{1,1} \cdot b_{0,2} \end{bmatrix}.$$

Example (Continued.)

Formally, the tensor product of matrices is a function

$$\otimes : \mathbb{C}^{m \times n} \times \mathbb{C}^{m' \times n'} \longrightarrow \mathbb{C}^{(m \cdot m') \times (n \cdot n')}$$

and is defined as

$$(A \otimes B)[j, k] = A[\lfloor j/m' \rfloor, \lfloor k/n' \rfloor] \cdot B[j \;\; Mod \;\; m', k \;\; Mod \;\; n'].$$

The unit of the monoidal structure is the 1 *and the unit morphism is the* 1 *by* 1 *identity matrix* [1].

Exercise

Prove that the Kronecker product of matrices respects matrix multiplication, i.e.,

$$(A_{m,n} \cdot A'_{n,p}) \otimes (B_{m',n'} \cdot B'_{n',p'}) = (A_{m,n} \otimes B_{m',n'}) \cdot (A'_{n,p} \otimes B'_{n',p'}).$$

(Yet another instance of the ubiquitous interchange law. See Important Categorical Idea.)

While we discuss many variations of strict monoidal categories in Chapter 7, there is one variation that arises with many of our examples, and it pays to describe it now.

Definition

A strict monoidal category $(\mathbb{A}, \otimes, I)$ *is* **strictly symmetric** *if for all objects a and b in $\mathbb{A}$, we have $a \otimes b = b \otimes a$, and for all morphisms f, g, we have $f \otimes g = g \otimes f$.*

Example

- *Of the examples of strict monoidal categories that we described, every commutative monoid can be thought of as a strictly symmetric, strict monoidal category.*
- *The strict monoidal categories* $(\mathbf{N}, +, 0)$, $(\{1\}^*, \bullet, \emptyset)$, $(P, \wedge, 1)$, $(P, \vee, 0)$, *and* $(\mathbf{2}, \wedge, 1)$ *are all strictly symmetric.*
- *In contrast,* $(\{0,1\}^*, \bullet, \emptyset)$ *(where* $0 \bullet 1 \neq 1 \bullet 0$*) and both strict monoidal category structures on* $\mathbf{K}\mathbb{Mat}$ *are not strictly symmetric.*

Properties of Strict Monoidal Categories

We conclude this Section with a theorem about strict monoidal categories that are monoids.

Theorem

Let $(M, \star, e)$ be a monoid, thought of as a one-object category. If M has a strict monoidal category structure $(M, \square, I)$, then $\star = \square$, and the monoid is a commutative monoid. Another way to say this is that every one-object strict monoidal category is a commutative monoid.

This theorem states that a set with two monoid structures that respect each other is a commutative monoid. The theorem goes by the name **Eckmann–Hilton argument**.

Properties of Strict Monoidal Categories

Proof.

First we show that the unit of the monoid is the same as the unit of the monoidal category.

$$\begin{aligned} e &= e \star e && e \text{ is a unit of the monoid} \\ &= (I \Box e) \star (e \Box I) && I \text{ is a unit of the monoidal category} \\ &= (I \star e) \Box (e \star I) && \text{from bifunctoriality} \\ &= I \Box I && e \text{ is a unit of the monoid} \\ &= I && I \text{ is a unit of the monoidal category.} \end{aligned}$$

□

Proof.

By the bifunctoriality of $\square$, we have that

$$(m \star n)\square(m' \star n') = (m\square m') \star (n\square n').$$

Setting $n = m' = e$ gives

$$(m \star e)\square(e \star n') = (m\square e) \star (e\square n'),$$

which reduces to $m\square n' = m \star n'$. That is, the two multiplications are the same. Setting $m = n' = e$ in the above Equation gives us

$$(e \star n)\square(m' \star e) = (e\square m') \star (n\square e).$$

which reduces $n\square m' = m' \star n$. However, since $\square = \star$ as operations we get that $n \star m' = m' \star n$. $\square$

Foreshadowing

- Chapter 5: Monoidal Categories
  - Section 5.2 Cartesian Categories
    - Cartesian Categories
    - Examples
    - co-Cartesian Categories
    - Examples

Introduction

- Many of the examples of combining objects and morphisms come from a category having a finite product structure.
- In general, such structures fail to be strictly associative.
- The simplest example is the category of sets with the Cartesian product.
- Given any sets S and T, there is a product $S \times T$.
- In general, for any three sets S, T and U, the product is not associative, i.e., $S \times (T \times U) \neq (S \times T) \times U$..
- The set on the left contains elements of the form $\langle s, \langle t, u \rangle \rangle$, while the set on the right contains elements of the form $\langle \langle s, t \rangle, u \rangle$.
- Although these sets are not equal, there is an isomorphism from one set to the other.

Introduction

- Even though finite products in a general category fail to be strictly associative.
- We saw in Chapter 3, that any category $\mathbb{A}$ with finite products entails that for all a, b, and c, there is an ismorphism $a \times (b \times c) \longrightarrow (a \times b) \times c$.
- This isomorphism is actually a component of a natural isomorphism from the functor

$$(\ ) \times ((\ ) \times (\ )) \colon \mathbb{A}^3 \longrightarrow \mathbb{A}$$

to the functor

$$((\ ) \times (\ )) \times (\ ) \colon \mathbb{A}^3 \longrightarrow \mathbb{A}.$$

- We also saw in Chapter 3 that if t is the terminal object in $\mathbb{A}$, then for every object a, there is an isomorphism $a \times t \longrightarrow a$.
- This isomorphism is a component of a natural isomorphism from the functor $(\ ) \times t \colon \mathbb{A} \longrightarrow \mathbb{A}$ to the identity functor $Id_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathbb{A}$.

- Furthermore, we showed us that the product structure induces a braid map $br\colon\ a \times b \longrightarrow b \times a$, which is an isomorphism.
- This isomorphism is a component of the a natural isomorphism from the functor $(\ \ ) \times (\ \ )$ to the functor $((\ \ ) \times (\ \ )) \circ br$.

Definitions

Let us formalize all of these notions about a category with a finite product.

Definition

A **Cartesian category** $(\mathbb{A}, \times, t)$ *is a category with finite products (the terminal object, t, is the product over the empty diagram.) The finite product structure induces the following natural isomorphisms.*

- *A way of reassociating the product: a* **reassociator** *natural isomorphism*

  $$\alpha\colon (\ ) \times ((\ ) \times (\ )) \Longrightarrow ((\ ) \times (\ )) \times (\ ).$$

  This means that for every a, b, and c, there is a component which is an isomorphism

  $$\alpha_{a,b,c}\colon a \times (b \times c) \longrightarrow (a \times b) \times c.$$

Definition (Continued.)

- *A way of eliminating the unit on the right: a* **right unitor** *natural isomorphism* $\rho \colon (\ \ ) \times t \Longrightarrow Id_{\mathbb{A}}$. *This means that for any* a, *there is a component which is an isomorphism* $\rho_a \colon a \times t \longrightarrow a$.
- *A way of eliminating the unit on the left: a* **left unitor** *natural isomorphism* $\lambda \colon t \times (\ \ ) \Longrightarrow Id_{\mathbb{A}}$. *This means that for any* a, *there is a component which is an isomorphism* $\lambda_a \colon t \times a \longrightarrow a$.
- *A way of reordering a product: a* **braiding** *natural isomorphism* $\gamma \colon (\ \ ) \times (\ \ ) \Longrightarrow ((\ \ ) \times (\ \ )) \circ br$. *This means that for every* a *and* b, *there is a component which is an isomorphism* $\gamma_{a,b} \colon a \times b \longrightarrow b \times a$.

These natural isomorphisms all interact and satisfy more axioms. For pedagogical reasons, we will resist listing these axioms for a little while.

Examples

There are many examples of Cartesian categories, some of which we have already seen.

Example

- $(\mathbb{Set}, \times, \{*\})$, $(\mathbb{Top}, \times, \{*\})$, *and* $(\mathbb{Manif}, \times, \{*\})$.
- *We are familiar with the product structure in* $\mathbb{Set}$.
- *The product structures in* $\mathbb{Top}$ *and* $\mathbb{Manif}$ *are less familiar. One must show that the Cartesian product of topological spaces has a topological structure, and this topological structure must conform with the projection functions. In detail, one must show that if* X *and* Y *are topological spaces, then so is* $X \times Y$, *and the projections* $\pi\colon X \times Y \longrightarrow X$ *and* $\pi\colon X \times Y \longrightarrow Y$ *are continuous maps.*
- *One must also show that the product of two manifolds is locally like* $\mathbf{R}^n$ *and that the projection functions are smooth.*
- *Tthe terminal object is the one-element set.*

Example

- $(\mathbb{Cat}, \times, \mathbf{1})$ *and* $(\mathbb{Graph}, \times, \mathbf{1})$.
- *We saw that* $\mathbb{Cat}$ *has a Cartesian category structure.*
- *We showed how to form the product of two categories.*
- *The unit is the terminal category that has a single object and a single identity morphism.*
- *We saw that the structure on the category of graphs is very similar.*

Examples

Example

- *The categories of most algebraic structures have a Cartesian category structure.*
- *In general, the product of two algebraic structures is the product of their underlying sets, and the operations are then performed pointwise, i.e. on each component.*
- *There is a prominent exception within the algebraic structures mentioned: the category of fields does not have products.*
- *Given two fields, F_1 and F_2, one can form the set of ordered pairs $F_1 \times F_2$. The pairs (a, b) can be added and multiplied*

$$(a, b)+(a', b') = (a+a', b+b') \quad (a, b) \cdot (a', b') = (a \cdot a', b \cdot b').$$

- *However, there are elements $(a, 0)$, which are not zero, and yet they do not have inverses. This means that the pairs fail to be a field. The obvious product in $\mathbb{Field}$ does not work.*

Example

$(\mathbf{K}\mathbb{FDV}\mathbb{ect}, \oplus, 0)$. *The category of finite dimensional* **K** *vector spaces and linear maps has a Cartesian category structure. There is a plethora of names for this operation:* **direct sum**, **product**, *and* **Cartesian product**.. *Since* $\mathbf{K}\mathbb{FDV}\mathbb{ect}$ *will play such an important role in this text, we discuss it in detail. If V and W are finite dimensional vector spaces, then*

$$V \oplus W = \{(v, w) : v \in V, w \in W\}$$

has a vector space structure. The vector space operations are defined as follows:

- *Addition is pointwise, i.e.,* $(v, w) + (v', w') = (v + v', w + w')$.
- *Scalar multiplication is pointwise, i.e.,* $c \cdot (v, w) = (c \cdot v, c \cdot w)$.
- *The zero is* $(0, 0)$.

The unit of this product is the trivial vector space 0.

The details of showing that $(\mathbf{K}\mathbb{Vect}, \oplus, 0)$ is a Cartesian category will be proven in the upcoming exercises.

Exercise

Show that the obvious projection maps

$$V \xleftarrow{\pi_V} V \oplus W \xrightarrow{\pi_W} W$$

defined by $(v, w) \mapsto v$ and $(v, w) \mapsto w$ are linear maps and they satisfy the universal property of being a product.

Examples

Exercise

Show that for three vector spaces V, W, and X, the vector space $V \oplus (W \oplus X)$ is isomorphic to $(V \oplus W) \oplus X$.

Exercise

Show that the trivial vector space 0 acts like a unit, i.e., show that $V \oplus 0$ and $0 \oplus V$ are isomorphic to V.

Exercise

Show that $V \oplus W$ is isomorphic to $W \oplus V$.

Examples

Here are two important theorems about the direct sum of vector spaces.

Theorem

For any finite dimensional vector spaces V and W,
$dim(V \oplus W) = dim(V) + dim(W)$.

Theorem

For any short exact sequence

$$0 \longrightarrow U \xrightarrow{T} V \xrightarrow{S} W \longrightarrow 0$$

V is isomorphic to $U \oplus W$.

Example

- *Any preorder category* $(P, \leq)$ *that has finite products has a Cartesian category structure.*
- *The product is called a* **meet** *and denoted* $\wedge$.
- *The unit is the terminal* 1 *and it satisfies the requirement that for all* p, *there is a unique isomorphism from* $p \wedge 1$ *to* p.
- *Notice that in contrast to where we discussed partial orders, for preorders* $p \wedge (q \wedge r)$ *need not be equal to* $(p \wedge q) \wedge r$.
- *Special cases of such preorder categories are the categories of propositions and predicates which we saw in the min-course on categorical logic. We showed that* $(\mathbb{Prop}, \wedge, True)$, *and for all* $\bar{x}$, $(\mathbb{Pred}(\bar{x}), \wedge, True)$ *are Cartesian categories.*

Examples

Example

$(\mathbb{CompFunc}, \times, *)$. *We saw that that the category of computable functions with the multiplication* $\times$ *and the terminal type* $*$ *is a Cartesian category.*

Examples

Example

The direct sum in $\mathbf{K}\mathbb{M}\mathrm{at}$ *has to be reexamined. We saw d that* $(\mathbf{K}\mathbb{M}\mathrm{at}, \oplus, 0)$ *has a strict monoidal category structure. In fact, it is an unusual example of a Cartesian category where the* $\oplus$ *is strictly associative. It pays to carefully show that the* $\oplus$ *monoidal structure satisfies the universal property of being a product in* $\mathbf{K}\mathbb{M}\mathrm{at}$*. The object* $m \oplus n = m + n$ *is the product of* m *and* n *with the projection given as*

$$m \xleftarrow{\begin{bmatrix} I_m & 0_{m,n} \end{bmatrix}} m + n \xrightarrow{\begin{bmatrix} 0_{n,m} & I_n \end{bmatrix}} n$$

Examples

Example (Continued.)

In order to see that these projections satisfy the universal properties, consider two morphisms (matrices) $A_{m,p} \colon p \longrightarrow m$ and $B_{n,p} \colon p \longrightarrow n$ as in:

$$\begin{tikzcd}
 & p \arrow[dl, "A_{m,p}"'] \arrow[d, "{\begin{bmatrix} A_{m,p} \\ B_{n,p} \end{bmatrix}}"] \arrow[dr, "B_{n,p}"] & \\
m & m+n \arrow[l, "{\begin{bmatrix} I_m & 0_{m,n} \end{bmatrix}}"'] \arrow[r, "{\begin{bmatrix} 0_{n,m} & I_n \end{bmatrix}}"'] & n
\end{tikzcd}$$

The $m+n$ by p matrix with $A_{m,p}$ on top of $B_{n,p}$ uniquely satisfies this property.

From duality, we know that whatever we said about the product and the terminal object also applies to the coproduct and the initial object.

Definition

A **co-Cartesian category** *or a* **coproduct category** $(\mathbb{A}, +, i)$ *or* $(\mathbb{A}, \amalg, i)$ *is a category with finite coproducts (the initial object, i, is the coproduct over the empty diagram.) The finite coproduct structure induces the following natural isomorphisms:*

- *A way of reassociating the coproduct: a* **reassociator** *natural isomorphism*

$$\alpha\colon (\ \ ) + ((\ \ ) + (\ \ )) \Longrightarrow ((\ \ ) + (\ \ )) + (\ \ ).$$

Definition

- *A way of eliminating the unit on the right: a* **right unitor** *natural isomorphism* $\rho\colon (\ \ ) + i \Longrightarrow Id_{\mathbb{A}}$.
- *A way of eliminating the unit on the left: a* **left unitor** *natural isomorphism* $\lambda\colon i + (\ \ ) \Longrightarrow Id_{\mathbb{A}}$.
- *A way of reordering a coproduct: a* **braiding** *or a* **cobraiding** *natural isomorphism* $\gamma\colon (\ \ ) + (\ \ ) \Longrightarrow ((\ \ ) + (\ \ )) \circ br$.

These natural isomorphisms all interact and satisfy more axioms. We will describe those axioms later.

Examples

There are many examples of co-Cartesian categories. Some of them we have already seen.

Example

$(\mathbb{Set}, +, \emptyset)$, $(\mathbb{Top}, +, \emptyset)$, *and* $(\mathbb{Manif}, +, \emptyset)$. *All three categories have coproduct structures. The empty set, the empty topological space, and the empty manifold, are the initial objects in their respective categories. One must show that in the cases of* $\mathbb{Top}$ *and* $\mathbb{Manif}$, *the coproduct exists, and the inclusion maps satisfy the universal properties and are continuous and smooth, respectfully.*

Example

$(\mathbb{Cat}, +, \mathbf{0})$ *and* $(\mathbb{Graph}, +, \emptyset)$. *The category* $\mathbb{Cat}$ *has a co-Cartesian category structure. The coproduct of two categories is their disjoint union. The unit is the empty category* $\mathbf{0}$. *The category of graphs has a similar structure.*

Example

$(Bool\mathbb{Func}, +, 0)$. *The objects are the natural numbers and the morphisms from m to n are all the set functions from the set* $\{0,1\}^m$ *to* $\{0,1\}^n$. *The tensor product on objects is addition of natural numbers. The morphisms is done as follows. Let* $f\colon \{0,1\}^m \longrightarrow \{0,1\}^n$ *and* $g\colon \{0,1\}^{m'} \longrightarrow \{0,1\}^{n'}$. *Then* $f+g\colon \{0,1\}^{m+m'} \longrightarrow \{0,1\}^{n+n'}$ *is defined by f the first m digits of the input and by g on the last* m' *digits.*

Example

- *A preorder category* $(P, \leq)$ *with finite coproducts has a co-Cartesian category structure where the coproduct is called* **join** *and denoted* $\vee$.
- *The initial object is denoted* 0 *and for all* p *there is an isomorphism* $p \vee 0 \longrightarrow p$.
- *In contrast to a partial order category, for an arbitrary preorder category,* $p \vee (q \vee r)$ *need not be equal to* $(p \vee q) \vee r$.
- *The categories we met in our mini-course in basic categorical logic,* $(\mathbb{Prop}, \vee, False)$ *and for all* $\bar{x}$ $(\mathbb{Pred}(\bar{x}), \vee, False)$, *are co-Cartesian categories.*

Exercise

We showed that $\mathbf{K}\mathbb{M}\mathrm{at}$ *has a strict monoidal structure and it has a Cartesian monoidal structure. Show that the product in* $\mathbf{K}\mathbb{M}\mathrm{at}$ *also makes it a co-Cartesian category. Use the fact that* $\mathbf{K}\mathbb{M}\mathrm{at}^{op} \cong \mathbf{K}\mathbb{M}\mathrm{at}$. *To see that the universal property of being a coproduct is satisfied, consider the following:*

$$\begin{tikzcd} m \arrow[r, "\begin{bmatrix} I_m \\ 0_{n,m} \end{bmatrix}"] \arrow[dr, "A_{p,m}"'] & m+n \arrow[d, "\begin{bmatrix} A_{p,m} B_{p,n} \end{bmatrix}"'] & n \arrow[l, "\begin{bmatrix} O_{n,m} \\ I_m \end{bmatrix}"'] \arrow[dl, "B_{p,n}"] \\ & p. & \end{tikzcd}$$

Examples

Exercise

We saw that $(\mathbf{K}\mathbb{Vect}, \oplus, 0)$ *is a Cartesian category. In fact it is also a co-Cartesian category. Show that the obvious inclusion maps*

$$V \xrightarrow{inc_V} V \oplus W \xleftarrow{inc_W} W$$

defined by $v \mapsto (v, 0)$ *and* $w \mapsto (0, w)$ *are linear maps and satisfy the universal properties of a coproduct. The* $\oplus$ *operation which is both a product and a coproduct is called a* **biproduct**.

Notice that some categories, such as $\mathbb{Set}, \mathbb{Top}, \mathbb{Manif}, \mathbb{Prop}, \mathbb{Cat}, \mathbb{Par}, \mathbf{K}\mathbb{Mat}$, and $\mathbf{K}\mathbb{Vect}$, have both Cartesian and co-Cartesian category structures.

Foreshadowing

- Chapter 5: Monoidal Categories
  - Section 5.3 Monoidal Categories
    - Definitions
    - Coherence Conditions
    - The Association Category
    - The Symmetry Category
    - Examples

Introduction

The structures we have seen till now, strict monoidal categories, Cartesian categories, and co-Cartesian categories, are special types of a more general structure that we will present here. By weakening the requirements, we get a notion that is more applicable (see Important Categorical Idea.)

Definition

A **monoidal category** $(\mathbb{A}, \otimes, I, \alpha, \lambda, \rho)$ *has the following structure:*

- *A category* $\mathbb{A}$.
- *A way of combining objects and morphisms: a bifunctor called the* **tensor product** *or the* **monoidal product** $\otimes\colon \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$.
- *A special object* I *of* $\mathbb{A}$ *called the* **unit**.

Definition

- *A way of reassociating the monoidal product: a natural isomorphism called a* **reassociator**

$$\alpha \colon (\ \ ) \otimes ((\ \ ) \otimes (\ \ )) \Longrightarrow ((\ \ ) \otimes (\ \ )) \otimes (\ \ ).$$

That is, for every a, b, and c, the component is an isomorphism

$$\alpha_{a,b,c} \colon a \otimes (b \otimes c) \longrightarrow (a \otimes b) \otimes c.$$

Definition

- *A way of eliminating the unit on the right: a natural isomorphism called a* **right unitor**

$$\rho\colon (\ \ )\otimes I \Longrightarrow Id_{\mathbb{A}}.$$

That is, for any a, the component is an isomorphism

$$\rho_a\colon a\otimes I \longrightarrow a.$$

- *A way of eliminating the unit on the left: a natural isomorphism* **left unitor**

$$\lambda\colon I\otimes(\ \ ) \Longrightarrow Id_{\mathbb{A}}.$$

That is, for any a, the component is an isomorphism

$$\lambda_a\colon I\otimes a \longrightarrow a.$$

Definition

The following requirements must be satisfied:

- *The reassociator must cohere with itself: for all objects a, b, c, and d the following* **pentagon coherence condition** *or* **Mac Lane's coherence condition** *or* **Stasheff's coherence condition**

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "id_a \otimes \alpha_{b,c,d}"] \arrow[d, "\alpha_{a,b,c\otimes d}"'] & a \otimes ((b \otimes c) \otimes d) \arrow[r, "\alpha_{a,b\otimes c,d}"] & (a \otimes (b \otimes c)) \otimes d \arrow[d, "\alpha_{a,b,c} \otimes id_d"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[rr, "\alpha_{a\otimes b,c,d}"'] & & ((a \otimes b) \otimes c) \otimes d
\end{tikzcd}$$

commutes.

(This is one of the most important diagrams in this class!)

Definition

- *The reassociator must cohere with the right and left unitors: for all objects a and b of $\mathbb{A}$, the following* **triangle coherence condition**

$$\begin{tikzcd} a \otimes (I \otimes b) \arrow[rr, "\alpha_{a,I,b}"] \arrow[dr, "id_a \otimes \lambda_b"'] & & (a \otimes I) \otimes b \arrow[dl, "\rho_a \otimes id_b"] \\ & a \otimes b & \end{tikzcd}$$

commutes.

While most of the variations of monoidal categories will be given in Chapter 7, there is one variation which deals with commutativity that arises so often that we provide it here.

Definition

A **symmetric monoidal category** $(\mathbb{A}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ *is a monoidal category* $(\mathbb{A}, \otimes, I, \alpha, \lambda, \rho)$ *with a natural isomorphism called a* **braiding** *that permutes two objects*

$$\gamma \colon \otimes \Longrightarrow \otimes \circ br.$$

That is, for objects a and b in $\mathbb{A}$ there is a natural isomorphism

$$\gamma_{a,b} \colon a \otimes b \longrightarrow b \otimes a.$$

Definition

In addition to being a monoidal category, a symmetric monoidal category must also satisfy the following conditions:

- *The braiding must be its own inverse: the* **symmetry coherence condition**

$$\begin{tikzcd} a \otimes b \arrow[rr, "\gamma_{a,b}"] \arrow[dr, equal] & & b \otimes a \arrow[dl, "\gamma_{b,a}"] \\ & a \otimes b. & \end{tikzcd}$$

Definitions

Definition

- *The braiding must cohere with itself and the associator: the* **hexagon coherence conditions**

$$
\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\alpha^{-1}_{a,b,c}"] \arrow[d, "\gamma_{a,b} \otimes Id_c"'] & a \otimes (b \otimes c) \arrow[r, "\gamma_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[d, "\alpha^{-1}_{b,c,a}"] \\
(b \otimes a) \otimes c \arrow[r, "\alpha^{-1}_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[r, "id_b \otimes \gamma_{a,c}"'] & b \otimes (c \otimes a)
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \otimes (b \otimes c) \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "id_a \otimes \gamma_{b,c}"'] & (a \otimes b) \otimes c \arrow[r, "\gamma_{a\otimes b,c}"] & c \otimes (a \otimes b) \arrow[d, "\alpha_{c,a,b}"] \\
a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"'] & (a \otimes c) \otimes b \arrow[r, "\gamma_{a,c} \otimes id_b"'] & (c \otimes a) \otimes b.
\end{tikzcd}
$$

Definition

- *The braiding must cohere with the left and right unitors: the* **triangle symmetry coherence condition**

$$
\begin{tikzcd}
a \otimes I \arrow[rr, "\gamma_{a,I}"] \arrow[dr, "\rho_a"'] & & I \otimes a \arrow[dl, "\lambda_a"] \\
& a. &
\end{tikzcd}
$$

Definitions

A monoidal category can be strictly associative and still be a (non-strictly) symmetric monoidal category.

Definition

A **strictly associative symmetric monoidal category** *is a symmetric monoidal category where α, λ, and ρ are all identity maps. In this case the coherence condition shorten to*

$$\begin{tikzcd} a \otimes b \otimes c \arrow[rr, "\gamma_{a,b\otimes c}"] \arrow[dr, "\gamma_{a,b}\otimes id_c"'] & & b \otimes c \otimes a \\ & b \otimes a \otimes c \arrow[ur, "id_b\otimes\gamma_{a,c}"'] & \end{tikzcd} \qquad \begin{tikzcd} a \otimes b \otimes c \arrow[rr, "\gamma_{a\otimes b,c}"] \arrow[dr, "id_a\otimes\gamma_{b,c}"'] & & c \otimes a \otimes b \\ & a \otimes c \otimes b. \arrow[ur, "\gamma_{a,c}\otimes id_b"'] & \end{tikzcd}$$

Examples

Let us explore some examples of monoidal categories.

Example

Every strict monoidal category is a monoidal category where the natural isomorphisms $\alpha, \rho,$ and λ are identity natural transformations. Obviously, all the coherence conditions are satisfied.

Examples

Example

- *Every Cartesian category and co-Cartesian category is a symmetric monoidal category.*
- *In Chapter 3, we showed that products induce reassociators, unitors, and braidings.*
- *We are left to show that the induced isomorphisms satisfy all the coherence conditions for a monoidal category.*
- *This is done within the text in three different ways. We leave it out of the slides because it is rather complex.*

Important Categorical Idea

Cartesian vs. Monoidal.

- *Although Cartesian categories are a special type of monoidal category, there could be significant differences between a Cartesian category and a general monoidal category.*
- *The tensor in a Cartesian category satisfies universal properties. This means that there are projection functions that satisfy all the universal properties.*
- *For example, in a Cartesian category, any object a, has a diagonal map $\Delta\colon a \longrightarrow a \times a$. In a general monoidal category, such a morphism need not exist.*
- *Also, in a Cartesian category, the monoidal product is automatically symmetric, while in a general monoidal category this is not necessarily true.*
- *This will have profound ramifications in the rest of this course.*

Example

There is a category called the **association category**, $\mathbb{A}\mathbb{S}\mathbb{S}\mathbb{O}\mathbb{C}$, *that is the paradigm of monoidal categories. The objects are the associations of letters. For example, the following are three associations:*

$$\bullet(\bullet\bullet) \qquad (\bullet(\bullet\bullet))(\bullet(\bullet(\bullet\bullet))), \qquad \bullet(\bullet(((\bullet)\bullet))\bullet)\bullet\,.$$

Each $\bullet$ *is a placeholder and only the arrangement of the parentheses are important. Some of the associations will have the letter* I *in some of their positions which will correspond to having a unit element in that position. For example:*

$$\bullet(I\bullet) \qquad (\bullet(I\bullet))(I(\bullet(\bullet I))), \qquad \bullet(I(((\bullet\bullet)I))\bullet)\bullet\,.$$

Example (Continued.)

The morphisms in this category are easy to describe: there is exactly one morphism between any two objects in the category. This makes the category $\mathbb{A}\mathrm{ssoc}$ *a contractible groupoid and a preorder category. The monoidal structure is also easy to describe: given associations* w *and* w'*, the tensor product is* $w \otimes w' = w \bullet w'$*. Given* $f\colon w \longrightarrow w'$ *and* $g\colon w'' \longrightarrow w^3$*, the tensor product* $f \otimes g$ *is the unique isomorphism* $w \bullet w'' \longrightarrow w' \bullet w^3$*. The unit of the monoidal structure is the association* I*. The uniqueness of the morphisms gives us the isomorphisms*

$$\alpha_{w,w',w''}\colon w \bullet (w' \bullet w) \longrightarrow (w \bullet w') \bullet w'.$$

The uniqueness of the morphisms also ensure that the pentagon and triangle coherence conditions are satisfied.
We will see in Chapter 6 how $(\mathbb{A}\mathrm{ssoc}, \otimes, I)$ *is the paradigm monidal category.*

Example

There is a category called the **symmetry category**, $\mathbb{S}\mathrm{ym}$, *that is the paradigm of symmetric monoidal categories. This category is built out of the symmetric groups, which we first define. For each natural number n, there is a* **symmetric group** *on n elements or the n-th symmetric group, written* S_n, *whose elements are ways of permuting the first n numbers. These groups are very important in all of mathematics and physics. A typical element of* S_6 *is*

$$(1,2,3,4,5,6) \longmapsto (3,4,1,6,5,2).$$

This permutation takes 1 *to* 3, 2 *to* 4, 3 *to* 1, *etc. Another typical element is the permutation*

$$(1,2,3,4,5,6) \longmapsto (5,3,6,4,2,1).$$

Example (Continued.)

These permutations can be composed by doing the first one and then the second one. The composition of these two permutations is the permutation that takes 1 *to* 3 *which further goes to* 6*, etc. Here is the composition:*

$$(1,2,3,4,5,6) \longmapsto (6,4,5,1,2,3).$$

There is an identity permutation which does not change anything, i.e.,

$$(1,2,3,4,5,6) \longmapsto (1,2,3,4,5,6).$$

Every permutation has an inverse. The inverse of the first permutation is

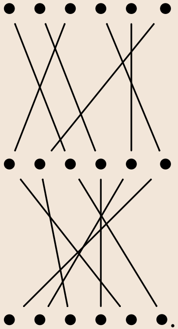
$$(1,2,3,4,5,6) \longmapsto (3,6,1,2,5,4).$$

Example (Continued.)

A graphical way of describing these permutations is having a line from the top number to the number it goes to. The permutations of the first and second example can be written as

Example (Continued.)

The composition of the first two examples can be seen as

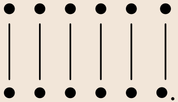


Follow the lines from the top to the bottom.

Examples

Example (Continued.)

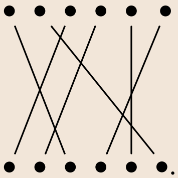
The identity permutation in S_6 can be viewed as



This corresponds to not changing anything. The composition of any permutation with the identity is the original permutation.

Example (Continued.)

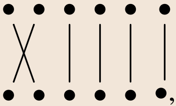
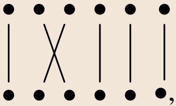
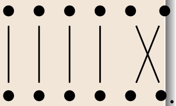
The inverse of the first example is its diagram turned upside-down:



Examples

Example (Continued.)

Using these graphical pictures, one can see that every permutation of the group can be constructed as a combination of small permutations that only change a number and its neighbor. We name these small permutations as $s_1, s_2, \ldots, s_{n-1}$. For S_6, these small permutations graphically look like this:

$s_1 =$  , $s_2 =$  , $\cdots$, $s_5 =$  .

Example (Continued.)

For the groups S_n, the generators $s_1, s_2, \ldots, s_{n-1}$ satisfy the following three equations:

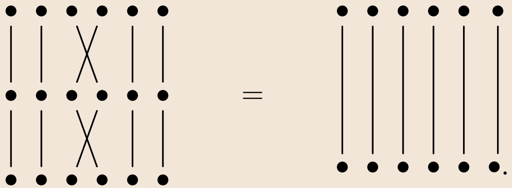
- $s_i s_i = e$
- $s_i s_j = s_j s_i$ for $|i - j| > 1$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Each of these equations can be understood graphically. We will show them in the next three slides.

Examples

Example (Continued.)

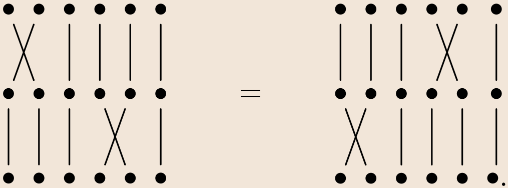
The meaning of the first equation is that switching the two numbers and then switching them again is like doing nothing. (or another way of saying this is that every generator is its own inverse.) This is the content of the following diagram



Examples

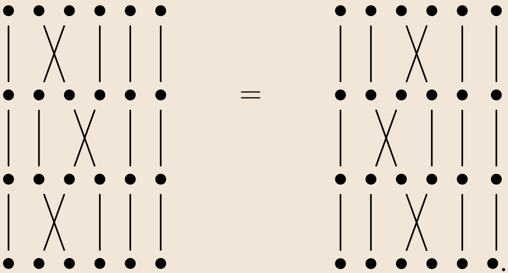
Example (Continued.)

The meaning of the second equation says that when the numbers being switched are more than one apart, then it does not matter in what order it is done. This corresponds to



Example (Continued.)

And finally, the third equation says that switching close numbers has the following property



Both diagrams describe the permutation
$(1,2,3,4,5,6) \longmapsto (1,4,3,2,5,6).$

Example (Continued.)

Now that we have these symmetry groups, let us gather them to form the category $\mathbb{S}\mathrm{ym}$. The objects are the natural numbers and the morphisms are given as follows

$$Hom_{\mathbb{S}\mathrm{ym}}(m,n) = \begin{cases} S_m & : \text{if } m = n \\ \emptyset & : \text{if } m \neq n. \end{cases}$$

One can envision this category as follows

$$\begin{array}{cccccc} S_0 & S_1 & S_2 & & S_n & \\ * & * & * & \cdots & * & \cdots \\ 0 & 1 & 2 & & n & \end{array}$$

Examples

Example (Continued.)

This category has a symmetric monoidal category structure. The monoidal product $\oplus$ on objects is addition, i.e., $m \oplus n = m + n$. On morphisms, $f\colon m \longrightarrow m$ and $g\colon n \longrightarrow n$, go to the function $f \oplus g\colon m + n \longrightarrow m + n$, where this function acts on each of its parts. Formally

$$(f \oplus g)(i) = \begin{cases} f(i) & : \text{if } i \leq m \\ g(i - m) + m & : \text{if } i > m. \end{cases}$$

We can visualize this as

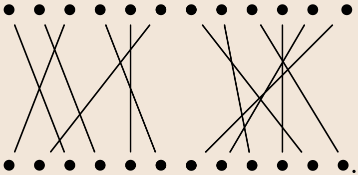
$$\begin{array}{ccc} m & \oplus & n \\ \downarrow f & & \downarrow g \\ m & \oplus & n. \end{array}$$

Example (Continued.)

In words, the $\oplus$ does both permutations without interfering with each other. For example, the monoidal product of the first two permutations is

$$(1,2,3,4,5,6,7,8,9,10,11,12) \longmapsto (3,4,1,6,5,2,11,9,12,10,8,7).$$

This can be visualized by putting the diagrams side-by-side:



It is easy to see that $\oplus$ is strictly associative.

Example (Continued.)

The interesting part is the braiding. For every m and n, there is an element in S_{m+n} *written* $\gamma_{m,n} : m+n \longrightarrow n+m$*, which is defined as*

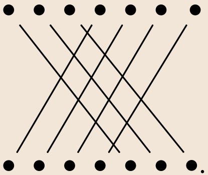
$$(1, \ldots m, m+1, \ldots m+n) \longmapsto (m+1, m+2, \ldots, m+n, 1, 2, \ldots, m).$$

Formally, this is

$$\gamma_{m,n}(p) = \begin{cases} p+m & : \text{if } p \leq m \\ p-m & : \text{if } p > m. \end{cases}$$

Example (Continued.)

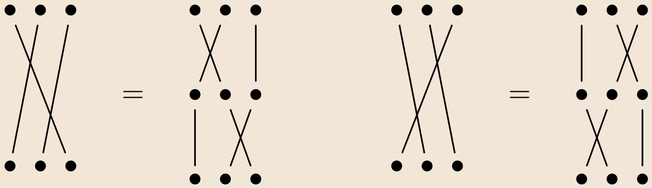
For example, if $m = 4$ *and* $n = 3$ *then* $\gamma_{4,3}$ *looks like*



Examples

Example (Continued.)

We are left to show that this braiding satisfies the two triangle coherence conditions. Given three integers m, n, and q, if we think of the m strands as one cable, the n strands as another cable, and the q strands as a third cable, then the two diagrams amount to



We will see in Chapter 6 how $(\mathbb{S}\mathrm{ym}, \oplus, \emptyset)$ *is the paradigm symmetric monidal category.*

Examples

Let us continue with our examples of monoidal categories.

Example

There is a monoidal category structure on $\mathbf{K}\mathbb{FDVect}$ *where the monoidal product is the tensor product of vector spaces that corresponds to the Kronecker product of matrices. This operation on vector spaces will be very important for quantum theory and quantum computing. The tensor product operation takes two vector spaces,* V *and* W*, and forms* $V \otimes W$*. If* $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ *is a basis for* V *and* $\mathcal{B}' = \{b'_1, b'_2, \ldots, b'_n\}$ *is a basis for* W*, then the basis for* $V \otimes W$ *will consist of vectors of the form*

$$\{b \otimes b' : b \in \mathcal{B}, b' \in \mathcal{B}'\}.$$

Example (Continued.)

A typical element of $V \otimes W$ will be a finite linear combination of these elements:

$$c_{1,1}(b_1 \otimes b_1') + c_{1,2}(b_1 \otimes b_2') + c_{1,3}(b_1 \otimes b_3') + \cdots + c_{m,n}(b_m \otimes b_n'.$$

The elements must satisfy a **bilinearity axiom**. *This says that the tensor product respects the addition in the two vector spaces. This amounts to*

$$(b + b') \otimes b'' = (b \otimes b'') + (b' \otimes b'')$$

and similarly,

$$b \otimes (b' + b'') = (b \otimes b') + (b \otimes b'').$$

Example (Continued.)

The bifunctor $\otimes$ *on the category* $\mathbf{K}\mathbb{FDV}\mathbb{ect}$ *is also defined on liner maps. If* $T\colon V \longrightarrow W$ *and* $T'\colon V' \longrightarrow W'$*, then there is a linear map* $T \otimes T'\colon V \otimes V' \longrightarrow W \otimes W'$*. This map is defined as*

$$(T \otimes T')(v \otimes w) = T(v) \otimes T'(w).$$

The unit of the monoidal category structure is the one dimensional vector space $\mathbf{K}$*. The basis for this vector space is the set* $\{1\}$*. The basis for* $V \otimes \mathbf{K}$ *is* $\{b_1 \otimes 1, b_2 \otimes 1, \ldots, b_m \otimes 1\}$*.*
The braiding is important. While $V \otimes W$ *is not equal to* $W \otimes V$*, there is an isomorphism between the two. Basically the isomorphism is induced by the map that takes the basis element* $b \otimes b'$ *to* $b' \otimes b$*. This braiding satisfies all the conditions of a symmetric monoidal category.*

We saw how the dimension of direct product of vector spaces work with addition. Here is a theorem about the dimension of the tensor product of vector spaces.

Theorem

For finite dimensional vector spaces V and W, we have $dim(V \otimes W) = dim(V) \cdot dim(W)$.

Proof.

A basis for the tensor product is the Cartesian product of the original basis, i.e.,

$$|\{b \otimes b' : b \in \mathcal{B}, b' \in \mathcal{B}'\}| = |\mathcal{B}| \cdot |\mathcal{B}'|.$$

$\square$

Example

$(\mathbb{Rel}, \otimes, \{*\})$. *The category* $\mathbb{Rel}$ *of sets and relations between them form a symmetric monoidal category. The value of* $\otimes$ *on sets* S *and* T *is* $S \times T$. *For relation* $Q\colon S \not\longrightarrow S'$ *and* $R\colon T \not\longrightarrow T'$, *we form*

$$Q \otimes R\colon S \times T \not\longrightarrow S' \times T'$$

which is defined as

$$((s,t),(s',t')) \in Q \otimes R \quad \text{if and only if} \quad (s,s') \in Q \text{ and } (t,t') \in R.$$

This is also written as

$$(s,t) \sim (s',t') \qquad \text{if and only if} \qquad s \sim s' \text{ and } t \sim t'.$$

Example (Continued.)

Given three sets S, T, and U, the reassociator $\alpha_{S,T,U}\colon S \otimes (T \otimes U) \not\longrightarrow (S \otimes T) \otimes U$ is defined as

$$((s,(t,u)),((s,t),u)) \in (S \otimes (T \otimes U) \times (S \otimes T) \otimes U)$$

or

$$(s,(t,u)) \sim ((s,t),u).$$

This reassociator clearly satisfies the pentagon condition.
The unit object is the one-element set $\{*\}$ and the right unitor on the set S is defined as $(s,*) \sim s$. There is a similar relation for the left unitor.
The braiding for sets S and T is $\gamma_{S,T}\colon S \otimes T \longrightarrow T \otimes S$ which is defined as $(s,t) \sim (t,s)$. This braiding satisfies the symmetry condition and the hexagons.

Examples

Endomorphisms of the unit of a monoidal category will be important. First a theorem.

Theorem

For I, the unit of a monoidal category, we have $\rho_I = \lambda_I \colon I \otimes I \longrightarrow I$.

Theorem

The endomorphisms of the unit of a monoidal category form a commutative monoid under composition. This means that for any pair $f \colon I \longrightarrow I$ and $g \colon I \longrightarrow I$, we have that $f \circ g = g \circ f \colon I \longrightarrow I$. Furthermore, we have that $f \otimes g = g \otimes f \colon I \otimes I \longrightarrow I \otimes I$.

Sometimes the endomorphisms of the unit have a lot more structure than just a commutative monoid.

Remark

In the monoidal category of complex vector spaces $(\mathbf{C}\mathbb{FDV}\mathbb{ect}, \otimes, \mathbf{C})$*, the set of linear maps from* $\mathbf{C}$ *to* $\mathbf{C}$ *is — not only a commutative monoid but it is — the field of complex numbers. By linearity, every map* $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ *is totally determined by the value* $f(1)$ *because* $f(c) = f(c \cdot 1) = c \cdot f(1)$.

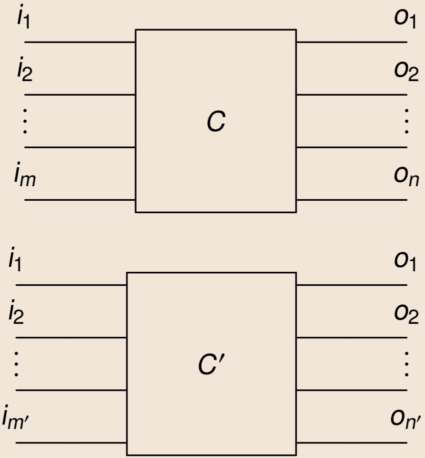
Examples

Example

- $(\mathbb{Circuit}, \oplus, \emptyset)$.
- *The category of logical circuits has a monoidal category structure. The tensor product is the disjoint union. That is, given two circuits, C with m input wires and n output wires, and C' with m' input wires and n' output wires, we can compose them to form $C \oplus C'$ as in the figure on the next slide.*
- *This new circuit has $m + m'$ input wires and $n + n'$ output wires.*
- *The unit is the empty circuit with no input wires and no output wires.*
- *We will see that this category is* **not** *a symmetric monoidal category.*

Examples

Example (Continued.)



Parallel composition of two circuits.

Foreshadowing

- When dealing with such a monoidal category, one must be concerned with a lot of "baggage" like the reassociator, the units, the coherence conditions, etc.
- This is in sharp contrast to a strict monoidal category which does not have a lot of "baggage."
- We will see in the next chapter that monoidal categories have a special relationship with strict monoidal categories.
- This relationship will help us easily deal with all the "baggage."

Foreshadowing

- Chapter 5: Monoidal Categories
  - Section 5.4 Coherence Theory
    - Introduction
    - Motivation
    - Shapes
    - Catalan Numbers

- Let us spend a few minutes meditating on coherence conditions. What are they all about?
- In order to get an intuition about operations and their axioms, we return to basic arithmetic. Consider the real numbers and an operation $\otimes$ that satisfies the associativity axiom: $a \otimes (b \otimes c) = (a \otimes b) \otimes c$. The $+$ and $\times$ operations satisfy this requirement.
- If this single axiom is satisfied, we are assured that no matter how many numbers and no matter how they are associated (bracketed), there will be a unique final value of the expression.

Introduction

- To reiterate, this means that for any numbers a, b, c, d, e, f, and g, and any two associations, their values will be equal, e.g.,

  $$(((a \otimes b) \otimes (c \otimes d)) \otimes e) \otimes (f \otimes g) = (a \otimes (b \otimes c)) \otimes ((d \otimes (e \otimes f)) \otimes g).$$

  In such a case, omit the parentheses: $a \otimes b \otimes c \otimes d \otimes e \otimes f \otimes g$.
- In contrast, if the associativity axiom is not satisfied, as with the $-$ or $\div$ operation, then different associations can give different values.

- Another example in basic arithmetic is commutativity.
- We ask if an operation $\otimes$ satisfies the commutativity axiom $a \otimes b = b \otimes a$.
- If the axiom is satisfied, we are assured that no matter how many numbers and no matter how they are ordered, there will be a unique final value. To reiterate, this means that for any numbers and for any two ordering of the numbers, their values are equal, e.g.,

$$c \otimes f \otimes a \otimes b \otimes g \otimes d \otimes e = g \otimes a \otimes b \otimes f \otimes c \otimes d \otimes e$$

This implies that the order of the elements is irrelevant.

- In contrast, with operations that do not satisfy the commutativity axiom, such as $-$ and $\div$, different orders of the numbers give different values.

Motivation

- Now let us return to categories.
- If a bifunctor $\otimes$ satisfies a strict associativity axiom, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, or a strict commutativity axiom, $a \otimes b = b \otimes a$, then these categories will satisfy the same properties of basic arithmetic.
- However, the universe is not always so pretty, and many categories and bifunctors do not have strict associativity or strict commutativity.

Motivation

- What can we say about monoidal categories when they do not satisfy strict axioms?
- In a monoidal category there are two functors $(\ )\otimes((\ )\otimes(\ ))$ and $((\ )\otimes(\ ))\otimes(\ )$ with a natural isomorphism between them. In other words, there is an isomorphism $\alpha\colon a\otimes(b\otimes c)\longrightarrow(a\otimes b)\otimes c$.
- We now ask what about functors that accept more letters and represent associations. How many natural isomorphisms are there between

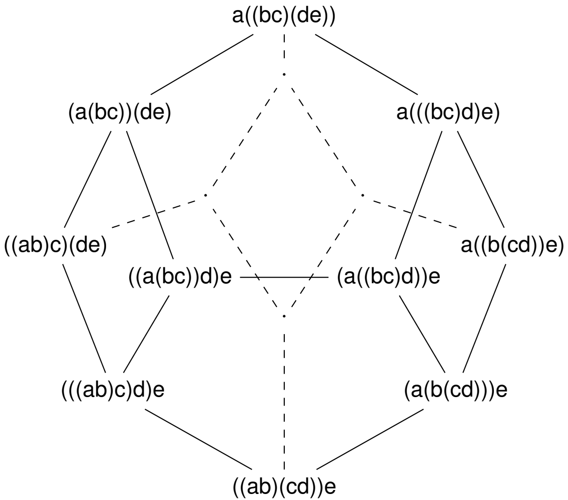
$$(((a\otimes b)\otimes(c\otimes d))\otimes e)\otimes(f\otimes g) \quad \text{and} \quad (a\otimes(b\otimes c))((d\otimes(e\otimes f))\otimes g)?$$

- For four letters, there are five ways of associating the letters, and the pentagon shows us that there are five different isomorphisms between them.
- How many isomorphisms are there from $a \otimes (b \otimes (c \otimes d))$ to $((a \otimes b) \otimes c) \otimes d$?
- If we assume that pentagon commutes then there is exactly one such isomorphism.
- We can think of this geometrically as follows. Before we assume that the pentagon commutes, there is a ring of isomorphisms. Once we assume the pentagon commutes, then think of the ring as a filled in disk. In that case, there is a unique path (map) from any functor to any other.

Motivation

- What if one has more than four letters?
- In order to get a handle on this, let us look at the case where there are five letters. For five letters, there are 14 ways of associating the letters. They are partially depicted in the next slide.
- The shape forms a sphere.

Motivation

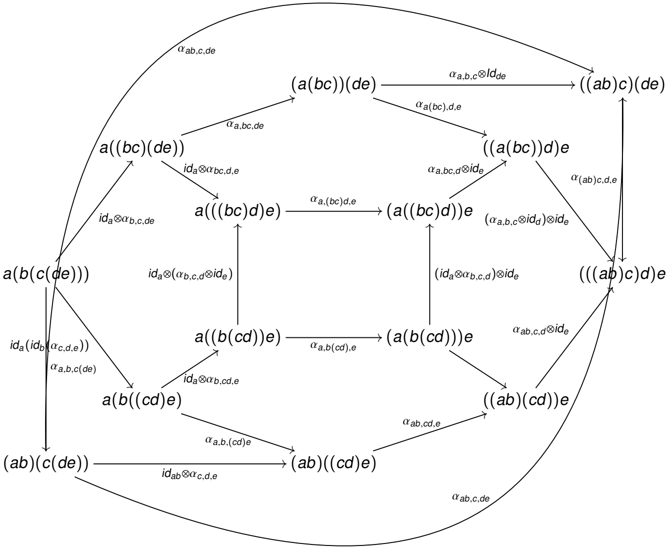


a((bc)(de))
(a(bc))(de)
a(((bc)d)e)
((ab)c)(de)
a((b(cd))e)
((a(bc))d)e
(a((bc)d))e
(((ab)c)d)e
(a(b(cd)))e
((ab)(cd))e

Motivation

- In order to see this shape more clearly, let us cut open the sphere and spread it out as in the next slide.
- Notice that the two long curved arrows from the lower-left corner to the upper right corner are the same map.
- This shape is made of 14 vertices and 21 arrows which form three squares and six pentagons.
- The squares are all naturality squares which commute because α is a natural transformation.
- The pentagons are all instances of the Mac Lane pentagon condition.

Motivation



(a(bc))(de)
((ab)c)(de)
a((bc)(de))
((a(bc))d)e
a(((bc)d)e)
(a((bc)d))e
a(b(c(de)))
(((ab)c)d)e
a((b(cd))e)
(a(b(cd)))e
a(b((cd)e)
((ab)(cd))e
(ab)(c(de))
(ab)((cd)e)
α_{ab,c,de}
α_{a,b,c}⊗Id_{de}
α_{a(bc),d,e}
α_{a,bc,de}
id_a⊗α_{bc,d,e}
α_{a,bc,d}⊗id_e
α_{(ab)c,d,e}
id_a⊗α_{b,c,de}
α_{a,(bc)d,e}
(α_{a,b,c}⊗id_d)⊗id_e
id_a⊗(α_{b,c,d}⊗id_e)
(id_a⊗α_{b,c,d})⊗id_e
α_{ab,c,d}⊗id_e
id_a(id_b(α_{c,d,e}))
α_{a,b,c(de)}
α_{a,b(cd),e}
id_a⊗α_{b,cd,e}
α_{ab,cd,e}
α_{a,b,(cd)e}
id_{ab}⊗α_{c,d,e}
α_{ab,c,de}

Motivation

- Here is the main point: if one assumes that α is a natural transformation, and that the pentagon commutes, then the entire diagram commutes.
- Rather than there being many morphisms from one vertex to another vertex, the entire diagram commutes and there is exactly one isomorphism made of α's between any two vertices.
- This is similar to what we saw with elementary arithmetic: if a single axiom is satisfied, then there is exactly one value.

Motivation

- What about six or more letters?
- It turns out that for any amount of letters, the figure is made of pentagons and squares.
- If you are working in a situation where there is coherence, then the pentagons commute.
- The squares commute from naturality.
- This means that with coherence and naturality, the whole shape commutes.

- In category theory there is always a goal of there being a unique isomorphism.
- We just proved that there is a unique isomorphism of a certain type in a monoidal category.
- Whenever we define a categorical structure, like an initial object or a product, we always prove that the object is "unique up to a unique isomorphism".
- What is this obsession with a *unique* isomorphism? Why can't there be more than one isomorphism between two objects in the category? Why do we make coherence conditions to ensure that there is a unique morphism between two vertices?

Motivation

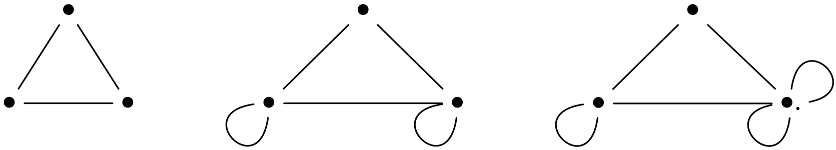
As we saw in Important Categorical Idea, there is a hierarchy:

- unique.
- unique up to a unique isomorphism.
- unique up to an isomorphism.

When two structures are isomorphic, it shows that they have the same structure. The isomorphisms are ways of showing that they are the same structure. However, when there is a unique isomorphism, then there is a unique way of reordering so that the two structures are the same.

Motivation

A simple example is in order. Consider the three graphs



- The graph on the left has three vertices and $3! = 6$ isomorphisms from it to itself.
- An isomorphism of the middle graph to itself must map the two bottom vertices to themselves or each other. There are only two maps that do this.
- In contrast, the graph on the right has exactly one (the trivial identity) isomorphism from it to itself. This makes the definition of the graph on the right more unique, since one cannot reshuffle the objects in a non-trivial way.

Motivation

- Mac Lane's pentagon condition is the beginning of a branch of category theory called **coherence theory**.
- In arithmetic and algebra, we study how different operations relate to each other. For example, the rule $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ says that the multiplication operation distributes over the addition operation.
- In category theory we go one level higher and study how different operations, which are represented by functors, relate to each other with natural transformations.
- With arithmetic and algebra, we are interested in determining if there is a single unique value.
- With coherence theory, we are concerned with determining the relationship between instances of the functors.

Motivation

- We study how these natural transformations "cohere" with themselves and with each other.
- In Mac Lane's pentagon condition, we see what happens when the reassociator coheres with itself.
- After we introduce functors between monoidal categories we will describe certain coherence theorems that explain properties of monoidal categories, and the relationship of general monoidal categories to strict monoidal categories.
- Since coherence theory deals with the relationship of morphisms between operations, as opposed to relationships between operations, it is sometimes called **higher-dimensional algebra**.
- Later, when discussing higher category theory, we will meet even higher coherence conditions.

Shapes

- In order to better understand coherence theory, we examine certain shapes or finite categories with the operations and the morphisms between them.
- The objects of the category correspond to the associations and the morphisms correspond to the reassociations.
- There is a sequence of such categories denoted $A_1, A_2, \ldots, A_n, \ldots$ where each one is called an **associahedron** (which is similar to the word "polyhedron," but related to the word "association") and together they are called **associahedra**.

Shapes

- In detail, the objects correspond to associations written as functors and the morphisms correspond to reassociations written as natural isomorphisms. Since they are isomorphisms, the categories are, in fact, groupoids.
- Here are the associahedra:
  - A_1. For one letter, there is exactly one object: a.
  - A_2. For two letters there is exactly one object: $a \otimes b$.
  - A_3. For three letters, there are two ways of associating them and an isomorphism between them:

$$a \otimes (b \otimes c) \longrightarrow (a \otimes b) \otimes c.$$

  - A_4. For four letters there are five different ways of associating them and there are five instances of isomorphism. This is depicted in the Mac Lane coherence condition . Notice that the pentagon is the same shape as a two-dimensional circle.

Shapes

- Some more associahedra:
  - A_5. For five letters, we saw diagrams here and here. Notice that this is a three-dimensional sphere.
  - A_n. What about for six or more letters? The shapes get too complicated to draw, but mathematicians know a lot of about them. For n letters there are $n-1$ monoidal products between the letters and there are

$$C_{n-1} = \frac{\binom{2(n-1)}{n-1}}{n} = \frac{(2(n-1))!}{(n)!(n-1)!}$$

    ways of associating or bracketing the letters. These numbers are called the **Catalan numbers**. The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796. The fact that the number of bracketings is equal to the Catalan numbers is left for the end of this Section. For n letters, the shape of the whole category is an $n-2$ dimensional sphere.
- The main point for all the associahedra is that they are made of commuting naturality squares and Mac Lane pentagons.

- What about commutativity? What shapes are formed if one looks at letters which can be permuted?
- Let us assume for a moment that we are dealing with strictly associative monoidal structure so we do not have to worry about parentheses.
- We will form categories (again groupoids) $P_1, P_3, P_3, \ldots$ each called a **permutohedron** and together are called the **permutahedra** which describe the permutations of the monoidal product.
  - P_1. For one letter, there is only one way of combining it: a.
  - P_2. For two letters, there are two ways of combining them with a braiding between them: $ab \longrightarrow ba$.

Shapes

- Some more permutahedra:
  - P_3. For three letters, there are $3! = 6$ ways of combining them. Some of the morphisms in P_3 looks like this:

$$\begin{tikzcd}
a \otimes b \otimes c \arrow[r, "id_a \otimes \gamma_{b,c}"] \arrow[d, "\gamma_{a,b} \otimes id_c"'] \arrow[dr, "\gamma_{a,b\otimes c}"'] & a \otimes c \otimes b \arrow[r, "\gamma_{a,c} \otimes id_b"] \arrow[dr, "\gamma_{a,c\otimes b}"] & c \otimes a \otimes b \arrow[d, "id_c \otimes \gamma_{a,b}"] \\
b \otimes a \otimes c \arrow[r, "id_b \otimes \gamma_{a,c}"'] & b \otimes c \otimes a \arrow[r, "\gamma_{b,c} \otimes id_a"'] & c \otimes b \otimes a.
\end{tikzcd}$$

    The two triangles commute because of the coherence conditions, and the middle quadrilateral commutes out of naturality.
  - P_n. For $n > 3$ letters, the categories get too complicated to draw. The shapes consist of $n!$ vertices. These shapes are made out of naturality squares and hexagons.

Shapes

There are categories that take into account associativity and commutativity. We denote the shapes as $AP_1, AP_2, AP_3, \ldots$ where each AP_n is called a **permuto-associahedron** and the collection is called the **permuto-associahedra**.

- AP_1. For one letter, there is only one vertex: a.
- AP_2. For two letters, there are two ways of ordering them with a braiding between them $ab \longrightarrow ba$.
- AP_3. For three letters there are 6 permutations and for each permutation, there are 2 ways of associating. This gives 12 objects and is partially depicted on the next slide:

Shapes

$$
\begin{tikzcd}
& a(bc) \arrow[r, "\alpha"] \arrow[dl, "id\otimes\gamma"'] \arrow[dddrrr, "\gamma_{a,bc}"] & (ab)c \arrow[r, "\gamma\otimes id"] & (ba)c \arrow[dr, "\alpha^{-1}"] & \\
a(cb) \arrow[d, "\alpha"'] \arrow[dddrrr, "\gamma_{a,cb}"'] & & & & b(ac) \arrow[d, "id\otimes\gamma"] \\
(ac)b \arrow[d, "\gamma\otimes id"'] & & & & b(ca) \arrow[d, "\alpha"] \\
(ca)b \arrow[dr, "\alpha^{-1}"] & & & & (bc)a \arrow[dl, "\gamma\otimes id"] \\
& c(ab) \arrow[r, "id\otimes\gamma"'] & c(ba) \arrow[r, "\alpha"'] & (cb)a. &
\end{tikzcd}
$$

Shapes

This diagram commutes because the middle rectangle is a naturality square. The other two hexagons are both instances ofthe coherence condition.

- AP_n. For $n > 3$ letters, the shapes get too big and complicated to draw. How many objects are there is in AP_n? There are $n!$ ways of ordering them, and for each ordering, there are C_{n-1} bracketings. This gives us $\dfrac{(2n-2)!}{(n-1)!}$ objects. The first few are 1, 2, 12, 120, 1680, 30240, 665280, 17297280.

Shapes

In Chapter 8 we will see that each of the collections of associahedra, permutahedra, and permuto-associahedra forms a categorical structure called an **operad**. We will meet these ideas again and see more ramifications of the coherence conditions.

Catalan numbers

In the book, the end of the Section proves the following.

Theorem

The number of legal bracketings of n letters is the $n-1$ Catalan number.

Its worth looking at the proof.

Foreshadowing

- Chapter 5: Monoidal Categories
  - Section 5.5 String Diagrams

Introduction

String diagrams are ways of describing the flow of morphisms within a category. We will introduce the concepts here and then build on them throughout the rest of the text. As the categorical constructions get more and more sophisticated and complicated, the string diagrams will start having all types of “bells and whistles.” We will meet many of these extra features in Chapter 7.

Introduction

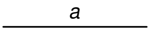
There is no consistency within the literature as to what direction the string diagrams should go.

- There are those inspired by physicists who have their diagrams go from bottom to top, similar to the Feynmen diagrams.
- Some authors have their diagrams go from top to bottom. This seems more natural as we read from top to bottom. We will sometimes follow this convention when talking about braids and tangles, which we will meet in Chapter 7.
- Many researchers draw their diagrams from left to right. This saves space and is similar to the way classical and quantum circuits are drawn.

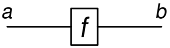
While string diagrams are important and helpful, our presentation will not be done exclusively with string diagrams. Alex Heller used to quip “A clear sentence is worth a thousand pictures.”

Basic String Diagrams

Since we began our journey, we have described an object in a category as a node a and morphism from one node to another as an arrow $a \xrightarrow{f} b$. String diagrams invert this convention. When discussing string diagrams, an object in a category corresponds to a line and the morphisms correspond to a box on the line that changes the line. So, the object a corresponds to

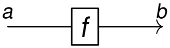


and a morphism $f\colon a \longrightarrow b$ corresponds to

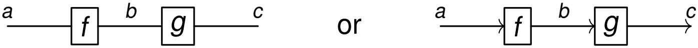


.

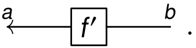
At times we stress the direction or orientation of an line by making the line into an arrow.



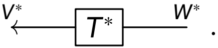
Composition of $f\colon a \longrightarrow b$ with $g\colon b \longrightarrow c$ is described as



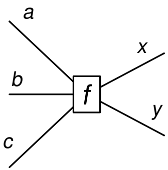
When there are arrows, one can describe a morphism going in the opposite direction, e.g., $f'\colon b \longrightarrow a$ as the string diagram



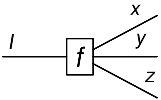
For example, the dual of a linear transformation $T^*\colon W^* \longrightarrow V^*$ is described by



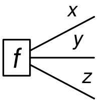
String diagrams get more interesting when dealing with monoidal categories. The morphism $f\colon a \otimes b \otimes c \longrightarrow x \otimes y$ is drawn as



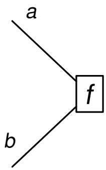
When the domain of a morphism is the unit of a monoidal category, such as $f\colon I \longrightarrow x \otimes y \otimes z$, rather than draw it as



we draw it as



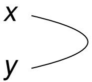
Similarly, a morphism $f\colon\ a \otimes b \longrightarrow I$ is drawn as



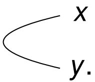
Of course a $f\colon\ I \longrightarrow I$ will be written as

Basic String Diagrams

In the event where the name of the morphism is not important, we draw the morphisms $x \otimes y \longrightarrow I$ and $I \longrightarrow x \otimes y$ as

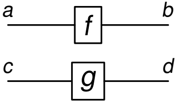


and



String Diagrams

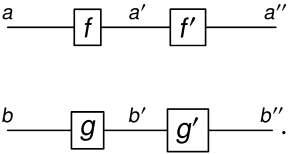
We are not only interested in string diagrams for the tensor product of objects. We form string diagrams for the tensor product of morphisms. The tensor product of $f\colon a \longrightarrow b$ and $g\colon c \longrightarrow d$ is



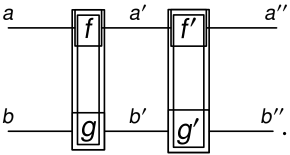
.

String Diagrams

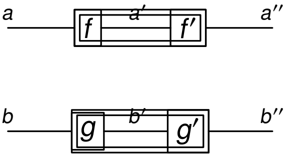
The interchange law (see Important Categorical Idea) should be viewed with a string diagram. It says, that four maps $f\colon a \longrightarrow a'$, $g\colon b \longrightarrow b'$, $f'\colon a' \longrightarrow a''$, and $g'\colon b' \longrightarrow b''$, as



can be correctly viewed in two different ways:

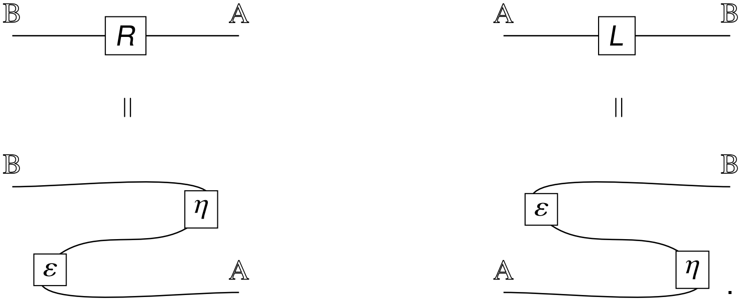


or as

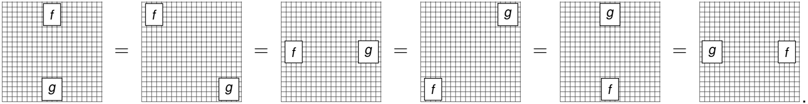


On the left is a sequential process of parallel processes, and on the right is a parallel process of sequential processes.

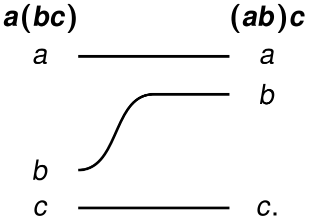
The triangle identities for an adjunction can be drawn as



The Theorem about that the endomorphisms of the unit of a monoidal category says that they are a commutative monoid. We can draw this as follows where the unit I and object $I \otimes I$ are depicted as the empty domain.

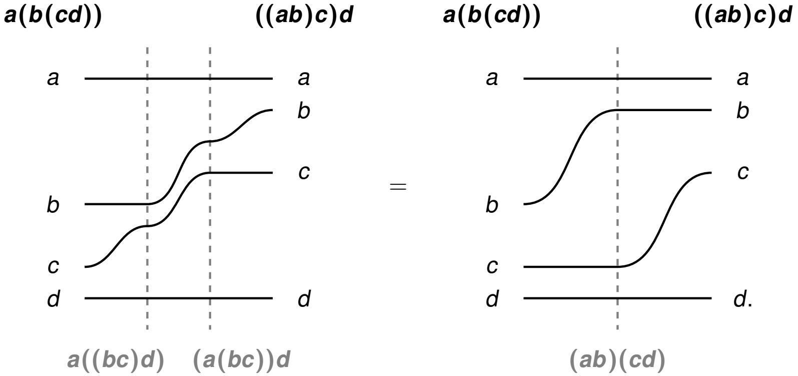


We can use these string diagrams to visualize the coherence conditions of a monoidal category. Let us consider $a \otimes b$ as two strings that are close to each other. In contrast, $(a \otimes b) \otimes c$ has a and b as close strings and c a further string. In this way, the reassociation α can be seen as a way of showing how strings change distances.



String Diagrams

The pentagon coherence condition can then be viewed as saying the following two string diagrams are equal



We will meet many more string diagrams in the coming pages.

Mini-course:

Advanced Linear Algebra

Foreshadowing

- Chapter 5: Monoidal Categories
  - Section 5.6: Mini-course: Advanced Linear Algebra
    - Hilbert Spaces
    - Operators on Hilbert Spaces
    - Eigenvalues and Eigenvectors

Here we go further with our study of linear algebra. There are many advanced parts of linear algebra, but we focus on what we need. In particular, the material we learn here will be central for our mini-courses on basic quantum theory, quantum computing, and for the rest of the text.

- Let us summarize what we know from our mini-course on basic linear algebra.
- The category $\mathbf{K}\mathbb{Vect}$ has $\mathbf{K}$ vector spaces as objects and linear maps as morphisms.
- We are mostly going to focus on the subcategory of finite dimensional $\mathbf{K}$-vector spaces $\mathbf{K}\mathbb{FDVect}$.
- Earlier in this chapter, we saw that $\mathbf{K}\mathbb{FDVect}$ has two distinct monoidal category structures: the Cartesian category structure $(\mathbf{K}\mathbb{FDVect}, \oplus, 0)$ called the direct product, and the monoidal category structure $(\mathbf{K}\mathbb{FDVect}, \otimes, \mathbf{K})$ called the tensor product.
- With every new notion, we will inquire how it respects these monoidal structures.

In this section we will focus our ideas and only discuss vector spaces over the complex numbers, $\mathbf{C}$, rather then over an arbitrary field $\mathbf{K}$. So, we will be looking at the category $\mathbf{C}\mathbb{V}\mathbb{ect}$ and $\mathbf{C}\mathbb{FDV}\mathbb{ect}$. We also look at complex vector spaces with more structure called Hilbert spaces.

Hilbert Spaces

In order to arrive at the definition of a Hilbert space, we must ramp up our knowledge of complex numbers and complex matrices.

Definition

If $c = a + bi$ is a complex number, then the **complex conjugate** *of c is $\overline{c} = a - bi$. This defines a functor $\overline{(\ \ )}: \mathbf{C} \longrightarrow \mathbf{C}$. Notice the complex conjugate operation is idempotent: $\overline{\overline{c}} = c$ and that if c is a real number (i.e., $c = a + 0i$,) then $\overline{c} = c$, i.e., the following commutes:*

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "\overline{(\ \ )}"] & & \mathbf{C} \\ & \mathbf{R}. \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Definition

If A is a matrix with complex entries, then we define the **conjugation** *operation, which we denote as $\overline{A}$, to be the complex matrix whose every entry is the complex conjugate of the original. Formally,*

$$\overline{A}[i,j] = \overline{A[i,j]}.$$

This is a functor $\overline{(\ )}\colon \mathbf{C}\mathbb{M}\mathrm{at} \longrightarrow \mathbf{C}\mathbb{M}\mathrm{at}$. Notice that conjugation is idempotent: $\overline{\overline{A}} = A$ and if A has only real entries, then $\overline{A} = A$, i.e., the following commutes:

$$\begin{tikzcd}
\mathbf{C}\mathbb{M}\mathrm{at} \arrow[rr, "\overline{(\ )}"] & & \mathbf{C}\mathbb{M}\mathrm{at} \\
& \mathbf{R}\mathbb{M}\mathrm{at}. \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Definition

If A is a matrix (with complex entries), then we define the **transpose** *operation, denoted as A^T, to be the complex matrix whose entries are flipped across the main diagonal of the original matrix. Formally,*

$$A^T[i,j] = A[j,i].$$

This is a functor $(\ )^T : \mathbf{C}\mathbb{M}\mathrm{at} \longrightarrow \mathbf{C}\mathbb{M}\mathrm{at}^{op}$. Notice that the transpose operation is idempotent: $(A^T)^T = A$.

Definition

We can combine the complex conjugation and the transpose to get the **adjoint** *or* **dagger** *operation . If A is a matrix with complex entries, then $A^\dagger = \overline{A}^T = \overline{A^T}$. Formally, $A^\dagger[i,j] = \overline{A[j,i]}$. In terms of categories, the fact that conjugation and transpose are idempotent means that they are both isomorphic functors. The $(\ )^\dagger : \mathbf{C}\mathbb{M}\mathrm{at} \longrightarrow \mathbf{C}\mathbb{M}\mathrm{at}^{op}$ functor is defined by composition as*

$$\begin{tikzcd}
\mathbf{C}\mathbb{M}\mathrm{at} \arrow[rr, "\overline{(\ )}"] \arrow[d, "T"'] \arrow[drr, "\dagger"] & & \mathbf{C}\mathbb{M}\mathrm{at} \arrow[d, "T"] \\
\mathbf{C}\mathbb{M}\mathrm{at}^{op} \arrow[rr, "\overline{(\ )}"'] & & \mathbf{C}\mathbb{M}\mathrm{at}.^{op}
\end{tikzcd}$$

Notice that dagger is idempotent: $(A^\dagger)^\dagger = A$ and hence an isomorphism. If A has only real entries, then $A^\dagger = A^T$.

Definition

The following commutative diagram expresses the relationship of real and complex matrices with the $\dagger$ operation

$$\begin{tikzcd} \mathbf{C}\mathbb{M}\mathrm{at} \arrow[r, "\dagger"] & \mathbf{C}\mathbb{M}\mathrm{at}^{op} \\ \mathbf{R}\mathbb{M}\mathrm{at} \arrow[u, hook] \arrow[r, "T"'] & \mathbf{R}\mathbb{M}\mathrm{at}^{op}. \arrow[u, hook] \end{tikzcd}$$

If A is an n by n matrix, then the **trace** *of A is the sum of the diagonal elements. That is,*

$$Tr(A) = \sum_{i=1}^{n} A[i,i].$$

This is a functor $Tr : \mathbf{C}^{n\times n} \longrightarrow \mathbf{C}$. In particular, the Trace of the identity matrix is equal to the dimension of the identity matrix.

Hilbert Spaces

Exercise

Show that complex conjugation respects complex addition and multiplication, i.e,

$$\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2} \qquad \text{and} \qquad \overline{c_1 \cdot c_2} = \overline{c_1} \cdot \overline{c_2}.$$

Exercise

Prove the following properties about conjugation, dagger, and trace.

Operation	*Conjugation*	*Dagger*	*Trace*
Matrix ad	$\overline{A+B} = \overline{A} + \overline{B}$	$(A+B)^\dagger = A^\dagger + B^\dagger$	$Tr(A+B) = TrA + TrB$
Scalar mlt	$\overline{c \cdot A} = \overline{c} \cdot \overline{A}$	$(c \cdot A)^\dagger = \overline{c} \cdot A^\dagger$	$Tr(c \cdot A) = c \cdot Tr(A)$
Matrix mlt	$\overline{A \cdot B} = \overline{A} \cdot \overline{B}$	$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$	$Tr(A \cdot B) = Tr(B \cdot A)$
Direct sm	$\overline{A \oplus B} = \overline{A} \oplus \overline{B}$	$(A \oplus B)^\dagger = A^\dagger \oplus B^\dagger$	$Tr(A \oplus B) = TrA + TrB$
Kronecker	$\overline{A \otimes B} = \overline{A} \otimes \overline{B}$	$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$	$Tr(A \otimes B) = TrA \cdot TrB$

Often we need the ability to compare vectors in a complex vector space. That is, we want a function that accepts two vectors and outputs a complex number telling us how they relate. A Hilbert space is a complex vector space with a comparing function that satisfies certain properties.

Definition

An **inner product space** *is a pair* $(V, \langle\ ,\ \rangle)$ *where*

- V *is a vector space, and*
- $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbf{C}$ *is a function called an* **inner product** *or a* **Hermitian inner product** *which is used to compare vectors.*

Definition (Continued.)

The inner product satisfies the following requirements:

- *A vector measured with itself is non-negative: for all v in V, $\langle v, v\rangle$ is a real number, $\langle v, v\rangle \geq 0$, and furthermore $\langle v, v\rangle = 0$ if and only if $v = 0$.*
- *The inner product respects addition in each variable :*

$$\langle v+v', w\rangle = \langle v, w\rangle+\langle v', w\rangle \quad \text{and} \quad \langle v, w+w'\rangle = \langle v, w\rangle+\langle v, w'\rangle.$$

- *The inner product is linear with scalar multiplication in the first variable and* **anti-linear** *with the second variable:*

$$\langle c \cdot v, w\rangle = c\langle v, w\rangle \qquad \text{and} \qquad \langle v, c \cdot w\rangle = \overline{c}\langle v, w\rangle.$$

- *The inner product is not symmetric, but* **skew-symmetric***:*

$$\langle v, w\rangle = \overline{\langle w, v\rangle}.$$

Example

Some examples of complex inner product spaces:

- $\mathbf{C}^n$. *The inner product is given as* $\langle v, w\rangle = v^\dagger w$.
- $\mathbf{C}^{m\times n}$. *The inner product is given as* $\langle A, B\rangle = Tr(A^\dagger B)$.
- $Func(\mathbf{N}, \mathbf{C})$. *The inner product is given as* $\langle f, g\rangle = \sum_{i=0}^{\infty} \overline{f(i)}g(i)$.
- $Func([a, b], \mathbf{C})$ *where* $[a, b] \subseteq \mathbf{R}$. *The inner product is given as* $\langle f, g\rangle = \int_a^b \overline{f(x)}g(x)dx$ *(when the integral exists.)*

We use the inner product to describe relationships between two vectors in a vector space. Two vectors v and w are called **orthogonal** if $\langle v, w\rangle = 0$.
The norm is a way of describing the length of a vector.

Definition

For a vector in a complex inner product space, the ***norm*** *of a vector is* $|v| = \sqrt{\langle v, v\rangle}$.

Example

For $\mathbf{C}^n$, *if* $v = [x_1, x_2, \ldots, x_n]^T$, *then* $|v| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Exercise

Show that the norm satisfies the following properties: for all v and w in V and for $c \in \mathbf{C}$

- *Norm is nondegenerate:* $|v| > 0$ *if* $v \neq 0$ *and* $|0| = 0$.
- *Norm satisfies the triangle inequality:* $|v + w| \leq |v| + |w|$.
- *Norm respects the scalar multiplication:* $|c \cdot v| = |c| \cdot |v|$.

With the notion of a norm, we can define special types of linear maps.

Definition

Let V and W be complex inner product space with norms $|\ |_V$ and $|\ |_W$ respectively. A linear map $T\colon V \longrightarrow W$ is **bounded** *if T does not stretch or shrink a vector too much. In detail, for T, there is a constant $r_T > 0 \in \mathbf{R}$ that depends on T, such that for all $v \in V$ we have*

$$|T(v)|_W \leq r_T|v|_V.$$

We will not be very bothered with bounded linear maps because we will mostly deal with finite dimensional vector spaces and linear maps between them are always bounded.

With the notion of inner product and a norm, we can define special types of bases of a vector space.

Definition

A basis $\mathcal{B} = \{b_1, b_2, b_3, \ldots\}$ *is called*

- **orthogonal** *if any two different vectors in the basis are orthogonal, that is, for any* $i \neq j$*, we have* $\langle b_i, b_j \rangle = 0$*.*
- **normal** *if the norm of every vector in the basis is 1, that is, for all* i*,* $\langle b_i, b_i \rangle = 1$*.*
- **orthonormal** *if it is both orthogonal and normal, that is, for all* i *and* j*,* $\langle b_i, b_j \rangle = \delta_{i,j}$ *where* $\delta_{i,j} = 1$ *if* $i = j$ *and* 0 *if* $i \neq j$*. The function* $\delta_{i,j}$ *is called the* **Kronecker delta***.*

With a norm, we can define a distance function. The intuition is that $d(v, w)$ is the length between the end of vector v and the end of vector w.:

Definition

Let $(V, \langle\ \ ,\ \ \rangle)$ *be a complex inner product space. We define a* **distance function**

$$d(\ \ ,\ \ ) : V \times V \longrightarrow \mathbf{R}$$

where

$$d(v_1, v_2) = |v_1 - v_2| = \sqrt{\langle v_1 - v_2, v_1 - v_2 \rangle}.$$

Exercise

Show that the distance function has the following properties for all $v, w, x \in V$*:*

- *Distance is nondegenerate:* $d(v, w) > 0$ *if* $v \neq w$ *and* $d(v, v) = 0$.
- *Distance satisfies the triangle inequality:* $d(v, x) \leq d(v, w) + d(w, x)$.
- *Distance is symmetric:* $d(v, w) = d(w, v)$.

Let us use the inner product to describe vectors in a vector space. Let V be a finite dimensional orthonormal basis $\mathcal{B} = \{b_1, b_2, b_3, \ldots, b_n\}$. Consider an arbitrary element $v = k_1 b_1 + k_2 b_2 + \cdots + k_n b_n$. The inner product of v with an element of the basis is

$$\langle k_1 b_1 + k_2 b_2 + \cdots + k_n b_n, b_i \rangle$$

which by linearity reduces to

$$\langle k_1 b_1, b_i \rangle + \langle k_2 b_2, b_i \rangle + \cdots + \langle k_n b_n, b_i \rangle.$$

By orthonormality, all but one of the terms are 0, and the entire expression is $\langle k_i b_i, b_i \rangle = k_i$. This means that when we compare v with b_i we get the scalar multiple in the b_i direction. This fact can be used to express v as

$$v = \langle v, b_1 \rangle + \langle v, b_2 \rangle + \cdots + \langle v, b_n \rangle = \Sigma_i \langle v, b_i \rangle$$

which will be very helpful.

We need the notion of a sequence of vectors getting closer and closer together.

Definition

Let $(V, \langle \quad , \quad \rangle)$ *be an inner product space. With the norm and a distance function we can go on to define special sequences. A* **Cauchy sequence** *is a sequence of vectors* $v_0, v_1, v_2, \ldots$ *such that for every* $\epsilon > 0$*, there exists an* $N_0 \in \mathbf{N}$ *with the property that*

$$\text{for all } m, n \geq N_0, \quad d(v_m, v_n) \leq \epsilon.$$

What happens when we take a limit of a Cauchy sequence?

Definition

A complex inner product space is called **complete** *if for any Cauchy sequence of vectors* $v_0, v_1, v_2, \ldots,$ *there exists a vector* $\hat{v} \in V$ *such that*

$$lim_{n\to\infty}|v_n - \hat{v}| = 0.$$

The intuition behind this is that a vector space with an inner product is complete if any sequence of vectors that gets closer and closer will eventually converge to a point.

Definition

A **Hilbert space** *is a complex inner product space that is complete. The category of Hilbert spaces and linear bounded maps between them is denoted* $\mathbb{Hilb}$*. The central focus will be the subcategory of finite dimensional Hilbert spaces and (bounded) linear maps between them denoted* $\mathbb{FDHilb}$*.*

Completeness might seem like an overly complicated (calculus) type of notion. Fear not! All the inner product spaces that we will meet will be complete and hence will be Hilbert spaces. In particular, every inner product on a finite dimensional complex vector space is automatically complete, hence every finite-dimensional complex vector space with an inner product is automatically a Hilbert space.

There is an obvious inclusion functor from $\mathbb{FDHilb}$ to $\mathbb{Hilb}$ and there are also obvious forgetful functors from categories of Hilbert spaces to categories of complex vector spaces. This can be summarized by the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{CVect} & \mathbb{Hilb} \arrow[l, two heads, "U"'] \\
\mathbb{CFDVect} \arrow[u, hook] & \mathbb{FDHilb}. \arrow[l, two heads, "U"] \arrow[u, hook]
\end{tikzcd}
$$

The notion of direct product and tensor product of vector spaces extends to a direct product and tensor product of Hilbert spaces. Let $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ be two Hilbert spaces. The direct product of the Hilbert spaces is $(V \oplus W, \langle\ ,\ \rangle_{V\oplus W})$ where the inner product is defined as

$$\langle (v, w), (v', w') \rangle_{V\oplus W} = \langle v, v' \rangle_V + \langle w, w' \rangle_W.$$

The tensor product of the Hilbert spaces is a completion of the space generated by $(V \otimes W, \langle\ ,\ \rangle_{V\otimes W})$ where the inner product is defined as

$$\langle (v \otimes w), (v' \otimes w') \rangle_{V\otimes W} = \langle v, v' \rangle_V \cdot \langle w, w' \rangle_W.$$

For the most part, we will work with finite dimensional Hilbert spaces and will not require this completion. It can be shown that these functions satisfy all the requirements of being an inner product space. Furthermore, the completeness of each of the inner product operators ensures that the combined inner products are also complete. Thus we have shown that the category of Hilbert spaces have two monoidal category structures $(\mathbb{Hilb}, \oplus, 0)$ and $(\mathbb{Hilb}, \otimes, \mathbf{C})$, which we call the direct product and the tensor product, respectively. Similarly, there are monoidal category structures $(\mathbb{FDHilb}, \oplus, 0)$ and $(\mathbb{FDHilb}, \otimes, \mathbf{C})$.

Operators on Hilbert Spaces

An operator on a Hilbert space is a linear map in $\mathbb{Hilb}$ ($\mathbb{FDHilb}$) from a Hilbert space to itself. We are interested in two special types of operators within the category: Hermitian operators and unitary operators.

First we take a brief detour and deal with matrices.

Definition

A real n by n matrix A is

- **symmetric** *if* $A^T = A$, *and*
- **orthogonal** *if* $A^T = A^{-1}$, *that is* $A^T \cdot A = Id_n = A \cdot A^T$.

A matrix is orthogonal if all its rows are orthogonal to each other (and each row is of norm 1) and all its columns are orthogonal to each other (and each column is of norm 1). By multiplying $A \cdot A^T$ *we see all the rows multiplied with each other. By multiplying* $A^T \cdot A$ *we see all the columns multiplied with each other. If both results are the identity, then the matrix is orthogonal (in fact it is orthonormal). A unitary matrix is the complex version of this.*

Definition

A complex n by n matrix A is

- **Hermitian** *if $A^{\dagger} = A$, and*
- **unitary** *if $A^{\dagger} = A^{-1}$ that is $A^{\dagger} \cdot A = Id_n = A \cdot A^{\dagger}$.*

How do the matrix operations respect these sets of matrices?

Theorem

The set of Hermitian matrices is closed under matrix addition, direct sum, and Kronecker product. In general, the set of Hermitian matrices is not closed under scalar multiplication and matrix multiplication.

Proof.

- Matrix addition. If A and B are both Hermitian, then $(A+B)^\dagger = A^\dagger + B^\dagger = A + B$ is also Hermitian.
- Scalar multiplication. If A is Hermitian and c is an arbitrary complex number, then $(c \cdot A)^\dagger = \overline{c} \cdot A^\dagger = \overline{c} \cdot A \neq c \cdot A$. So it is not closed.
- Matrix multiplication. If A and B are both Hermitian, then

  $$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger = B \cdot A \neq A \cdot B.$$

  So it is not closed.
- Direct sum. If A and B are both Hermitian, then $(A \oplus B)^\dagger = A^\dagger \oplus B^\dagger = A \oplus B$.
- Kronecker product. If A and B are both Hermitian, then $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$.

$\square$

Theorem

The set of unitary matrices is closed under matrix multiplication, direct sum, and Kronecker product. In general, the set of unitary matrices is not closed under matrix addition and scalar multiplication.

Proof.

- Matrix addition. If A and B are both unitary, then

$$(A+B)^{\dagger}\cdot(A+B) = (A^{\dagger}+B^{\dagger})\cdot(A+B) = A^{\dagger}A+A^{\dagger}B+B^{\dagger}A+B^{\dagger}B$$

$$= Id + A^{\dagger}B + B^{\dagger}A + Id.$$

  This, in general, is not equal to the identity. So it is not closed under this operation.

- Scalar multiplication. If A is unitary and c is an arbitrary complex number, then
  $(c \cdot A)^{\dagger} \cdot (c \cdot A) = \overline{c} \cdot A^{\dagger} \cdot (c \cdot A) = \overline{c} \cdot c \cdot A^{\dagger} \cdot A = \overline{c} \cdot cId \neq Id.$
  So it is not closed.

- Matrix multiplication. If A and B are both unitary, then

$$(A \cdot B)^{\dagger} \cdot (A \cdot B) = B^{\dagger} \cdot A^{\dagger} \cdot A \cdot B = B^{\dagger} \cdot Id \cdot B = B^{\dagger} \cdot B = Id.$$

  So it is closed.

Proof.

- Direct sum. If A and B are both unitary, then

  $$(A \oplus B)^{\dagger} \cdot (A \oplus B) = (A^{\dagger} \oplus B^{\dagger}) \cdot (A \oplus B) = (A^{\dagger} A) \oplus (B^{\dagger} B) = Id \oplus Id.$$

  So it is closed.
- Kronecker product. If A and B are both unitary, then

  $$(A \otimes B)^{\dagger} \cdot (A \otimes B) = (A^{\dagger} \otimes B^{\dagger}) \cdot (A \otimes B) = (A^{\dagger} A) \otimes (B^{\dagger} B) = Id \otimes Id.$$

  So it is closed.

$\square$

Definition

Since unitary (orthogonal) matrices are closed under matrix multiplication, and the identity matrix is unitary (and orthogonal), the collection of unitary (and orthogonal) matrices form a subcategory of matrices which we denote as $\mathbb{UMat}$ *(*$\mathbb{OMat}$*).*

In contrast, Hermitian and symmetric matrices are not closed under matrix multiplication and do not form a subcategory of matrices. We symbolize the collection of Hermitian and symmetric matrices as collections of arrows $\{Hermitian\}$ and $\{symmetric\}$

Definition

$$\begin{tikzcd}
\mathbb{OMat} \arrow[r, hook] \arrow[d, hook] & \mathbb{UMat} \arrow[d, hook] \\
\mathbf{R}\mathbb{Mat} \arrow[r, hook] & \mathbf{C}\mathbb{Mat} \\
\{\textit{symmetric}\} \arrow[r, hook] \arrow[u, hook] & \{\textit{Hermitian.}\} \arrow[u, hook]
\end{tikzcd}$$

Since unitary matrices are closed under direct product and tensor product, they form strict monoidal category structures: $(\mathbb{UMat}, \oplus, 0)$ *and* $(\mathbb{UMat}, \otimes, 1)$.

Let us move the discussion from matrices to operators.

Definition

Let V be a Hilbert space and let $T\colon V \longrightarrow V$ be a bounded linear map. An **adjoint** *or* **dagger** *of T is a unique function $T^\dagger\colon V \longrightarrow V$ that satisfies the following equation for all $v, w \in V$:*

$$\langle T(v), w\rangle = \langle v, T^\dagger(w)\rangle.$$

(The reader should see a resemblance of this definition to the definition of adjoint functors. In fact, adjoint functors got their name because they are similar to an adjoint linear map.)

Exercise

Prove that the adjoint linear map satisfies the following properties with respect to operations on linear transformations

Operation	*Dagger*
Addition	$(T+T')^\dagger = T^\dagger + T'^\dagger$
Scalar mult.	$(c \cdot T)^\dagger = \overline{c} \cdot T^\dagger$
Composition	$(T \circ T')^\dagger = T'^\dagger \circ T^\dagger$
Direct sum	$(T \oplus T')^\dagger = T^\dagger \oplus T'^\dagger$
Tensor product	$(T \otimes T')^\dagger = T^\dagger \otimes T'^\dagger$

The adjoint is used to define special types of operators.

Definition

A bounded linear operator $T: V \longrightarrow V$ is

- **Hermitain** *or* **self-adjoint** *if it is its own adjoint, i.e., $T^{\dagger} = T$.*
- **unitary** *if its adjoint is its inverse, i.e, $T^{\dagger} = T^{-1}$. That is, $T^{\dagger} \circ T = Id_V = T \circ T^{\dagger}$.*

Notice that a unitary operator is, by definition, invertible.

Theorem

$T\colon V \longrightarrow V$ is Hermitian if and only if $\langle Tv, w\rangle = \langle v, Tw\rangle$.

Proof.

($\Longrightarrow$) Since $T^\dagger = T$.
($\Longleftarrow$) By the uniqueness of the adjoint. $\square$

Theorem

If $T\colon V \longrightarrow V$ is a unitary operator, then it preserves norms. That is, $|v| = |Tv|$.

Proof.

$$
\begin{aligned}
|v| &= \sqrt{\langle v, v\rangle} && \text{by definition of norm.}\\
&= \sqrt{\langle v, I(v)\rangle} && I \text{ is the identity operator}\\
&= \sqrt{\langle v, T^\dagger T v\rangle} && \text{because } T^\dagger T = I\\
&= \sqrt{\langle Tv, Tv\rangle} && \text{by definition of adjoint}\\
&= |Tv| && \text{by definition of norm.}
\end{aligned}
$$

$\square$

Exercise

Show that unitary operators are closed under composition, direct product, and tensor product.

Definition

Since unitary operators are closed under composition, and identity morphisms are unitary, the collection of Hilbert spaces and unitary operators form a subcategory $\mathbb{UHilb}$ *of* $\mathbb{Hilb}$*. Similarly, one can talk about the subcategory of finite dimensional Hilbert spaces and unitary operators* $\mathbb{UFDHilb}$*.*

Since all unitary operators are invertible, $\mathbb{UHilb}$ and $\mathbb{UFDHilb}$ are actually groupoids. There are inclusion functors from $\mathbb{UHilb}$ to $\mathbb{Hilb}$ and from $\mathbb{UFDHilb}$ to $\mathbb{FDHilb}$. Since unitary matrices are closed under direct sum and tensor product, there are monoidal category structures $(\mathbb{UHilb}, \oplus, 0)$, $(\mathbb{UFDHilb}, \oplus, 0)$, $(\mathbb{UHilb}, \otimes, \mathbf{C})$, and $(\mathbb{UFDHilb}, \otimes, \mathbf{C})$.

Let us relate Hermitian and unitary matrices with Hermitian and unitary linear operators.

Theorem

- *If A is a Hermitain matrix, then T_A is Hermitian operator.*
- *Let V be a finite dimensional Hilbert space with a basis $\mathcal{B}$ and a Hermitian operator $T\colon V \longrightarrow V$. Then there is a Hermitian matrix A such that for all v in V, $T(v) = Av$.*
- *If A is a unitary matrix, then T_A is a unitary operator.*
- *Let V be a finite dimensional Hilbert space with a basis $\mathcal{B}$ and a unitary operator $T\colon V \longrightarrow V$. Then there is a unitary matrix A such that for all v in V, $T(v) = Av$.*

With the last two statement of this theorem, we can prove that there is an equivalence of categories between $\mathbb{UMat}$ and $\mathbb{UFDHilb}$.

Operators on Hilbert Spaces

Let us summarize all the functors that we have been dealing with relating sets, vector spaces, matrices, and Hilbert spaces.

$$\begin{tikzcd}
\mathbb{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{C}\mathbb{Vect} \arrow[l, bend left, "U"] & \mathbb{Hilb} \arrow[l, two heads, "U"'] & \mathbb{UHilb} \arrow[l, hook'] \\
\mathbb{FinSet} \arrow[u, hook] \arrow[r, "F"] & \mathbf{C}\mathbb{FDVect} \arrow[u, hook] & \mathbb{FDHilb} \arrow[u, hook] \arrow[l, two heads, "U"'] & \mathbb{UFDHilb} \arrow[u, hook] \arrow[l, hook'] \\
 & \mathbf{C}\mathbb{Mat} \arrow[u, hook, "\simeq"] & & \mathbb{UMat}. \arrow[u, hook, "\simeq"'] \arrow[ll, hook']
\end{tikzcd}$$

In this section we deal with linear maps from a complex vector space to itself. Sometimes such operators only change the magnitude of a vector and leave the direction fixed. That is, the operator only changes the vector by a scalar multiple. Those vectors are almost a fixed point of the operator. We will see that such vectors and the amount that they are changed by the operators are very important for our study of such operators.

Definition

For a linear map $T\colon V \longrightarrow V$, if there is a $v \in V$ and a $\lambda \neq 0$ in $\mathbb{C}$ such that

$$T(v) = \lambda \cdot v$$

then v is called an **eigenvector** *of T and λ is called the* **eigenvalue** *of v. (The word eigen is from the German "own" or "self.")*

For every $\lambda \in \mathbf{C}$, there is an operator $\lambda I \colon V \longrightarrow V$ that is defined for $v \in V$ as $\lambda I(v) = \lambda \cdot v$. It is not hard to see that λI is a linear map. Since the subtraction of two operators is still an operator, for any operator $T \colon V \longrightarrow V$, we have that $T - \lambda I$ is a linear map and is defined as $(T - \lambda I)(v) = T(v) - \lambda I(v) = T(v) - \lambda \cdot v$. For every eigenvalue λ, $T - \lambda I$ is a linear map and its kernel consists of those vectors that are eigenvectors for λ

$$V_\lambda = \{v \in V : T(v) = \lambda \cdot v\} \subseteq V$$

and is called the **eigenspace** of T belonging to λ. If V is a vector space of functions, then an eigenvector will be called an "eigenfunction." If a basis consists of eigenvectors for some operator, then the basis is called an **eigenbasis**.
From a categorical point of view, an eigenspace is simply the equalizer in the diagram

$$\begin{tikzcd} V_\lambda \arrow[r, hook, "inc"] & V \arrow[r, shift left, "\lambda I"] \arrow[r, shift right, "T"'] & V. \end{tikzcd}$$

What type of eigenvalues and eigenvectors do our two favorite operators have?

Theorem

The eigenvalues of a Hermitian operator are real.

Proof.

Let $T\colon V \longrightarrow V$ be a Hermitian operator and let v be a vector of norm 1, i.e. $\langle v, v\rangle = 1$. Say that λ is the eigenvalue of T, i.e., $T(v) = \lambda v$.

$$\begin{aligned}
\lambda &= \lambda\langle v, v\rangle && \text{because}\langle v, v\rangle = 1\\
&= \langle \lambda v, v\rangle && \text{by linearity of the inner product}\\
&= \langle Tv, v\rangle && \text{by definition of an eigenvalue}\\
&= \langle v, Tv\rangle && T \text{ is Hermitian}\\
&= \langle v, \lambda v\rangle && \text{by definition of an eigenvalue}\\
&= \bar{\lambda}\langle v, v\rangle && \text{by anti-linearity of the inner product}\\
&= \bar{\lambda} && \text{because } \langle v, v\rangle = 1.
\end{aligned}$$

Since $\lambda = \bar{\lambda}$, it is real. □

Theorem

The eigenvectors of distinct eigenvalues for a Hermitian operator are orthogonal.

Proof.

Let $T\colon V \longrightarrow V$ be a Hermitian operator and let v and w be vectors such that $T(v) = \lambda v$ and $T(w) = \mu w$ with $\lambda \neq \mu$.

$$\begin{aligned}
\lambda\langle v, w\rangle &= \langle \lambda v, w\rangle && \text{by linearity of the inner product}\\
&= \langle Tv, w\rangle && \text{by definition of an eigenvalue}\\
&= \langle v, Tw\rangle && T \text{ is Hermitian}\\
&= \langle v, \mu w\rangle && \text{by definition of an eigenvalue}\\
&= \bar{\mu}\langle v, w\rangle && \text{by anti-linearity of the inner product}\\
&= \mu\langle v, w\rangle && \mu \text{ is real.}
\end{aligned}$$

Since $\lambda\langle v, w\rangle = \mu\langle v, w\rangle$ and $\lambda \neq \mu$, it must be that $\langle v, w\rangle = 0$. □

Notice that the converse of this theorem is not necessarily true. That means that there could be a Hermitian operator with two eigenvectors that are not orthogonal but their respective eigenvalues are equal.

Eigenvalues and Eigenvectors

Theorem

The eigenvalues of a unitary operator have modulus 1.

Proof.

Let $T\colon V \longrightarrow V$ be a unitary operator and let v be an eigenvector with eigenvalue λ.

$$\begin{aligned}
|v|^2 &= |Tv|^2 && \text{bec } T \text{ preserves norms}\\
&= \langle Tv, Tv\rangle && \text{by definition of norm}\\
&= \langle \lambda v, \lambda v\rangle && \text{by definition of eigenvalue}\\
&= \lambda\langle v, \lambda v\rangle && \text{linear of the first variable}\\
&= \lambda\bar{\lambda}\langle v, v\rangle && \text{anti-linear of the second variable}\\
&= \lambda\bar{\lambda}|v|^2. && \text{definition of norm.}
\end{aligned}$$

So $|v|^2 = \lambda\bar{\lambda}|v|^2$ and by dividing out, we get that $|\lambda| = 1$. □