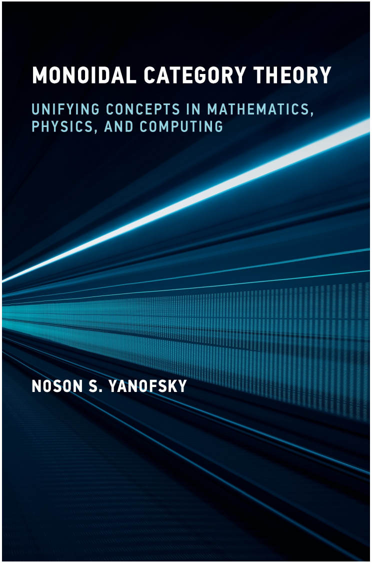
Monoidal Category Theory: Unifying concepts in Mathematics, Physics, and Computing



Chapter 4:

Relationships

Between Categories

Foreshadowing

- Chapter 4: Relationships Between Categories
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Category theory is about relating different categories. We will see that there are many possible relationships between categories. In this chapter we formally introduce functors between categories and natural transformations between functors. We then move on to employ these structures to relate categories in a myriad of ways.

Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.1: Functors
    - Definitions
    - Forgetful Functors
    - Properties of Functors
    - Isomorphic Functors
    - Contravariant Functos
    - Multifunctors
    - Examples

Introduction

- While categories are interesting by themselves, the true power of category theory is seeing the way categories are related to each other.
- Just as a function is the main way of expressing a relationship between sets, so too, a functor is going to be the main way of showing a relationship between categories.
- A functor assigns to an object of one category an object of another category and similar with morphisms.
- The assignment respects domains and codomains.
- We will show how to connect seemingly disparate areas using functors.

Definitions

Definition

Given two categories $\mathbb{A}$ and $\mathbb{B}$, a **functor** *F from $\mathbb{A}$ to $\mathbb{B}$, written $F\colon \mathbb{A} \longrightarrow \mathbb{B}$, is a rule that assigns to every object a of $\mathbb{A}$ an object $F(a)$ of $\mathbb{B}$, and assigns to every morphism $f : a \longrightarrow a'$ in $\mathbb{A}$ a morphism $F(f)\colon F(a) \longrightarrow F(a')$ in $\mathbb{B}$. These assignments must satisfy the following two requirements:*

- *Functors respect the compositions of morphisms: for $f\colon a \longrightarrow a'$ and $f'\colon a' \longrightarrow a''$ in $\mathbb{A}$, we require that $F(f' \circ_{\mathbb{A}} f) = F(f') \circ_{\mathbb{B}} F(f)$ where the $\circ_{\mathbb{A}}$ on the left is the composition in $\mathbb{A}$ while the $\circ_{\mathbb{B}}$ on the right is the composition in $\mathbb{B}$. (We omit such subscripts when they are clear from the contexts.)*
- *Functors respect identity morphisms, i.e., they take identity morphisms in one category to identity morphisms in the second category: for all a in $\mathbb{A}$, we require $F(id_a) = id_{F(a)}$ where id_a is in $\mathbb{A}$ while $id_{F(a)}$ is in $\mathbb{B}$.*

Definitions

A functor $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ can be thought of as a pair of functions $F_0 \colon Ob(\mathbb{A}) \longrightarrow Ob(\mathbb{B})$ and $F_1 \colon Mor(\mathbb{A}) \longrightarrow Mor(\mathbb{B})$. The fact that $f : a \longrightarrow a'$ in $\mathbb{A}$, must go to $F(f) \colon F(a) \longrightarrow F(a')$ in $\mathbb{B}$ means that the assignments respect the domain and codomain functions as in the commuting of the following two diagrams:

$$\begin{array}{ccc} Mor(\mathbb{A}) & \xrightarrow{F_1} & Mor(\mathbb{B}) \\ {\scriptstyle dom_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle dom_{\mathbb{B}}} \\ Ob(\mathbb{A}) & \xrightarrow[F_0]{} & Ob(\mathbb{B}) \end{array} \qquad \begin{array}{ccc} Mor(\mathbb{A}) & \xrightarrow{F_1} & Mor(\mathbb{B}) \\ {\scriptstyle cod_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle cod_{\mathbb{B}}} \\ Ob(\mathbb{A}) & \xrightarrow[F_0]{} & Ob(\mathbb{B}). \end{array}$$

These two commuting squares are similar to the diagrams we met in the definition of a graph homomorphism.

A functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ takes a morphism $a \longrightarrow a'$ in $\mathbb{A}$ to a morphism $F(a) \longrightarrow F(a')$ in $\mathbb{B}$, which means that for any a, a' in $\mathbb{A}$ there is a function of Hom sets:

$$Hom_{\mathbb{A}}(a, a') \longrightarrow Hom_{\mathbb{B}}(F(a), F(a')).$$

These functions of Hom sets will be of central importance in this course.

Examples

We begin with some simple examples of functors.

Example

Let $\mathbb{A}$ be any category, then there is an **identity functor** $Id_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}$ *which is defined for object a as $Id_{\mathbb{A}}(a) = a$ and defined similarly for morphisms in $\mathbb{A}$.*

Example

Consider the natural numbers $\mathbf{N}$ *as a partial order category. There is a functor $(\ \ ) + 5 \colon \mathbf{N} \longrightarrow \mathbf{N}$ that accepts a number m and adds five to it, i.e., $m + 5$. This is a functor because it respects morphisms (if $m \leq m'$ then $m + 5 \leq m' + 5$) and it respects the identity $(id_m) + 5 = id_{m+5}$.*

Example

There is a functor $\mathcal{P}\colon \mathbb{Set} \longrightarrow \mathbb{Set}$ *that takes a set* S *to its powerset* $\mathcal{P}(S)$*. A set function* $f\colon S \longrightarrow S'$ *will go to the set function* $\mathcal{P}(f)\colon \mathcal{P}(S) \longrightarrow \mathcal{P}(S')$ *which is defined for* $X \subseteq S$ *as its image under* f*. In other words,*

$$\mathcal{P}(f)(X) = f(X) = \{f(x) \in S' \mid x \in S\} \subseteq S'.$$

The function $\mathcal{P}(f)$ *is also denoted* f_* *and is called the* ***direct image*** *of* f*. The requirements for being a functor are easily seen to be satisfied. This functor is called the* ***direct image functor****.*

Any functor from a category to itself is called an **endofunctor**. The previous three examples are endofunctors.

Example

- *Consider the real numbers,* $\mathbf{R}$*, and the integers,* $\mathbf{Z}$*, each thought of as a partial order.*
- *There is a functor* $Floor\colon \mathbf{R} \longrightarrow \mathbf{Z}$ *that takes any real number* r *to* $Floor(r)$*, the greatest integer less than or equal to* r*.*
- *For example,* $Floor(3.7563) = 3$ *and* $Floor(-5.87) = -6$*.*
- *The assignment* $Floor$ *is a functor because if* $r \leq r'$*, then* $Floor(r) \leq Floor(r')$*. The floor functor is also denoted as* $Floor(r) = \lfloor r \rfloor$*.*
- *There is also a ceiling functor,* $Ceil\colon \mathbf{R} \longrightarrow \mathbf{Z}$ *that takes any real number,* r*, to the least integer greater or equal to* r*. The ceiling functor is denoted as* $Ceil(r) = \lceil r \rceil$*.*

Examples

Example

- *There is a functor* $D\colon \mathbb{S}\mathrm{et} \longrightarrow \mathbb{G}\mathrm{raph}$ *that takes every set* S *to the discrete graph* $D(S)$.
- *This is a graph with only objects and no morphisms.*
- *Given a set function* $f\colon S \longrightarrow S'$, *there is a similar graph homomorphism* $D(S) \longrightarrow D(S')$ *that takes objects of* $D(S)$ *to objects of* $D(S')$ *as described by* f.
- *The requirements of being a functor are easily seen to be satisfied.*

Forgetful Functors

Example

- *There is a* **forgetful functor** $U\colon \mathbb{Group} \longrightarrow \mathbb{Set}$ *from the category of groups to the category of sets that "forgets" the group structure.*
- *Remember that a group is a set G with extra structure, i.e,* $(G, \star, e, (\ )^{-1})$.
- *The functor U takes a group and forgets the rest of the structure, i.e., it takes a group to its underlying set.*
- *This means U performs the following operation:* $(G, \star, e, (\ )^{-1}) \mapsto G$.
- *A group homomorphism is a set function that respects all the structure.*

Forgetful Functors

Example (Continued.)

- *The functor U will take a group homomorphism to its underlying set function, i.e., a group homomorphism* $f\colon G \longrightarrow G'$ *will go to the set function* $U(F)\colon U(G) \longrightarrow U(G')$.
- *It is not hard to see that the requirements for U being a functor are satisfied.*
- *There is a similar functor from the category of monoids* $U\colon \mathbb{Monoid} \longrightarrow \mathbb{Set}$ *that forgets the monoid structure.*

Examples

The following two functors will be of importance in the coming sections.

Example

- *For every set B there is a functor $L_B\colon \mathbb{Set} \longrightarrow \mathbb{Set}$ that is defined on set A as $L_B(A) = A \times B$.*
- *Morphism $f\colon\ A \longrightarrow A'$ goes to $L_B(f) = f \times id_B\colon\ A \times B \longrightarrow A' \times B$.*
- *For every set B there is also a functor $R_B\colon \mathbb{Set} \longrightarrow \mathbb{Set}$ that is defined on input set C as $R_B(C) = Hom_{\mathbb{Set}}(B, C)$.*
- *For $f\colon\ C \longrightarrow C'$ we define $R_B(f)\colon Hom_{\mathbb{Set}}(B, C) \longrightarrow Hom_{\mathbb{Set}}(B, C')$ on input $g\colon\ B \longrightarrow C$ as $R_B(f)(g) = f \circ g\colon\ B \longrightarrow C \longrightarrow C'$ which is in $Hom_{\mathbb{Set}}(B, C')$.*
- *The functor R_B is called a* **representable functor***. It is represented by B.*

Example

- *There is a functor* $F\colon \mathbb{S}\mathrm{et} \longrightarrow \mathbb{M}\mathrm{onoid}$ *that takes every set* S *to the* **free monoid** $F(S) = S^*$.
- *This monoid consists of the set of all strings of elements in* S. *In other words, think of* S *as a set of letters or an alphabet, then* $F(S)$ *is the set of words that can be made from those letters.*
- *Given two words* $X = s_1 s_2 \cdots s_m$ *and* $Y = s_1' s_2' \cdots s_n'$, *the multiplication is the concatenation of the two strings* $X \bullet Y = s_1 s_2 \cdots s_m s_1' s_2' \cdots s_n'$.

Examples

Example (Continued.)

- *The unit of the monoid is the empty word* $\emptyset$ *that has no letters.*
- *For a set function* $f\colon S \longrightarrow T$ *the value of* $F(f)\colon F(S) \longrightarrow F(T)$ *takes a word in* S *such as* $s_1 s_2 \cdots s_m$ *to* $f(s_1)f(s_2)\cdots f(s_m)$ *which is a word in* T*.*
- *This functor is important in mathematics and is called the* **free monoid functor***.*
- *It is also important in computer science, where it is called the* **list functor***.*

Types of Functors

There are many different types of functors. With sets we ask if a function is one-to-one or onto. With categories we ask if a functor is one-to-one or onto with respect to objects and with respect to morphisms.

Types of Functors

There are properties of functors that depend on the way it assigns objects from one category to the other.

Definition

A functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ can be

- **injective on objects**
- **surjective on objects**
- **bijective on objects**.

There are also properties of functors that depend on morphisms. We call a functor

- **full** *if, for all a and a', the map* $Hom_{\mathbb{A}}(a, a') \longrightarrow Hom_{\mathbb{B}}(F(a), F(a'))$ *is onto*
- **faithful** *if for all a and a', the map* $Hom_{\mathbb{A}}(a, a') \longrightarrow Hom_{\mathbb{B}}(F(a), F(a'))$ *is one-to-one.*
- **fully faithful** *when a functor is both full and faithful.*

Types of Functors

Let us discuss various families or types of functors.

Definition

If $\mathbb{A}$ *is a subcategory of* $\mathbb{B}$ *then there is an* **inclusion functor** *from* $\mathbb{A}$ *to* $\mathbb{B}$ *that takes every object in* $\mathbb{A}$ *into the same object in* $\mathbb{B}$*. We denote such a functor as* $\mathbb{A} \hookrightarrow \mathbb{B}$*. Inclusion functors are injective on objects and faithful. We also say that such functors are the* **identity on objects***, that is, the functor takes an object* a *to* a*.*

Examples

Example

Examples of inclusion functors abound.

- *There is an inclusion of finite sets into sets,* $\mathbb{FinSet} \hookrightarrow \mathbb{Set}$.
- *Abelian groups include into all groups,* $\mathbb{AbGp} \hookrightarrow \mathbb{Group}$.
- *There are three related categories that have sets as objects. The category* $\mathbb{Set}$ *has set functions as morphisms,* $\mathbb{Par}$ *has partial functions as morphisms, and* $\mathbb{Rel}$ *has relations as morphisms. There are inclusion functors*

$$\mathbb{Set} \hookrightarrow \mathbb{Par} \hookrightarrow \mathbb{Rel}.$$

Notice that all three of these categories have the same objects, namely sets, and both inclusion functors are the identity on objects.

Examples

Example (Continued.)

- *If you think of the sets of numbers* $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, *and* $\mathbf{C}$ *with addition as monoids (one-object categories) then there are inclusion functors.*

$$\mathbf{N} \hookrightarrow \mathbf{Z} \hookrightarrow \mathbf{Q} \hookrightarrow \mathbf{R} \hookrightarrow \mathbf{C}.$$

  All these functors take the single object to the single object (hence they are technically bijective on objects) and faithful on the set of morphisms.

- *If you think of the numbers* $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, *and* $\mathbf{C}$ *as partial orders (in fact, all except the usual way of thinking about* $\mathbf{C}$ *are total orders,) then these inclusion functors are injective on objects and full and faithful on morphisms.*

Examples

Example (Continued.)

- *Remember that for a field* $\mathbf{K}$*, the category of* $\mathbf{K}\mathbb{M}\mathrm{at}$ *has the natural numbers as objects and for* $m, n \in \mathbf{N}$*,* $Hom_{\mathbf{K}\mathbb{M}\mathrm{at}}(m, n)$ *is the set of* n *by* m *matrices with entries in* $\mathbf{K}$*. We have the following inclusion functors*

$$\mathbf{N}\mathbb{M}\mathrm{at} \hookrightarrow \mathbf{Z}\mathbb{M}\mathrm{at} \hookrightarrow \mathbf{Q}\mathbb{M}\mathrm{at} \hookrightarrow \mathbf{R}\mathbb{M}\mathrm{at} \hookrightarrow \mathbf{C}\mathbb{M}\mathrm{at}.$$

  Notice that all these categories have the natural numbers as objects and all the functors are the identity on objects.
- *There is an inclusion of the category of partial orders into the category of preorders. Since both categories have order preserving maps as their morphisms, the inclusion* $\mathbb{PO} \hookrightarrow \mathbb{P}\mathrm{re}\mathbb{O}$ *is full and faithful.*
- *There is an inclusion for every* n *of* n-$\mathbb{M}\mathrm{anif} \hookrightarrow \mathbb{M}\mathrm{anif}$.

Definition

A special type of inclusion functor is an **embedding**. *This is a functor that is not only an inclusion (injective on objects and faithful) but also full. This means that between any two objects in the subcategory, the Hom sets are isomorphic. A subcategory with such an embedding is called a* **full subcategory**.

Examples

Example

The following are embeddings:

- $\mathbb{FinSet} \hookrightarrow \mathbb{Set}$
- $\mathbb{AbGp} \hookrightarrow \mathbb{Group}$
- $\mathbb{PO} \hookrightarrow \mathbb{PreO}$
- n-$\mathbb{Manif} \hookrightarrow \mathbb{Manif}$.

Types of Functors

We saw a functor that forgets the group structure of a group. There are many functors that are called **forgetful functors** which forget or disregard part of the structure of an object in a category. Such functors are usually denoted by the letter U, which might stand for "underlying." Since two different morphisms will go to two different underlying morphisms, forgetful functors are faithful.

Examples

Example

- *Similar to the forgetful functor* $U\colon \mathbb{Group} \longrightarrow \mathbb{Set}$, *there are forgetful functors from categories like* $\mathbb{Magma}$, $\mathbb{Monoid}$, $\mathbb{Ring}$, $\mathbb{Field}$, *etc., to* $\mathbb{Set}$.
- *We do not have to forget all the structure. For example, given a ring* $(M, \star, e, -, \odot, u)$, *we can forget its second operation and second unit to get an abelian group* $(M, \star, e, -)$. *This gives us a functor* $U\colon \mathbb{Ring} \longrightarrow \mathbb{AbGp}$. *We can further forget its inverse operation and get a monoid* $(M, \star, e)$. *This is a functor* $U\colon \mathbb{Ring} \longrightarrow \mathbb{Monoid}$. *Rather than forgetting the second operation, second unit, and inverse operation, we can forget its first operation, first unit, and inverse operation and get a monoid:* $(M, \odot, u)$. *This is a different functor* $U'\colon \mathbb{Ring} \longrightarrow \mathbb{Monoid}$. *There are many more examples along this path.*

Examples

Example (Continued.)

- *There is a functor* $\mathbb{Top} \longrightarrow \mathbb{Set}$ *that forgets the open set structure,* $U((T,\tau)) = T$. *There is also a forgetful functor that takes a manifold (which is a topological space with more structure) and forgets the manifold structure. The output is a topological space. This gives us a functor* $\mathbb{Manif} \longrightarrow \mathbb{Top}$. *Furthermore, we can forget the manifold structure "all the way down" to* $\mathbb{Set}$ *and get a forgetful functor* $\mathbb{Manif} \longrightarrow \mathbb{Set}$.
- *Given a complex scalar multiplication on a vector space, one can forget the action of the imaginary numbers and get a real scalar multiplication on the vector space. In other words, the scalar multiplication* $\cdot : \mathbf{R} \times V \longrightarrow V$ *is simply the restriction of* $\cdot : \mathbf{C} \times V \longrightarrow V$ *to the real numbers). This entails a forgetful functor* $\mathbf{C}\mathbb{Vect} \longrightarrow \mathbf{R}\mathbb{Vect}$.

In contrast to an inclusion functor, a forgetful functor is usually not injective on objects. For example, the forgetful functor from $\mathbb{Top}$ to $\mathbb{Set}$ is not injective on objects because there might be many different topologies that one can put on a single set. Similarly, the forgetful functor from graphs to sets is not injective on objects because there are many different graphs for a set of vertices.

Examples

Remark

This brings to light a general thought about the relationship between structures. Consider two types of structures: A-structures and B-structures. There are a lot of possible different relationships between the two structures. Let us highlight two possible relationships:

- *There can be an inclusion functor from the category of A-structure to the category of B-structure. This is the case when the A-structures are special types of B-structures that satisfy more requirements. Another way to say it is that A-structures have more properties than B-structures. For example, an abelian group is a special type of group which satisfies commutativity. Or a finite set is a special type of set that is finite. Or a n-dimensional manifold is a special type of manifold.*

Remark (Continued.)

- *There can be a forgetful functor from the category of A-structures to the category of B-structures. This is the case when the A-structures have more operations than the B-structures. Another way to say it is that A-structures have more structure than B-structures. For example, a ring has more operations than a group and hence there is a forgetful functor from rings to groups. Another example: a partial order has more structure than a set, so there is a forgetful functor from partial orders to sets. Yet another example: a topological space has more structure than a set, so there is a forgetful functor from topological spaces to sets.*

Remark (Continued.)

We can see this clearly with all the algebraic structures described in this definition. We write $\hookrightarrow$ *for an inclusion functor and* $\xrightarrow{U}$ *for a forgetful functor.*

$$\begin{array}{ccccccccc}
\mathbb{Field} & \xrightarrow{U} & \mathbb{Ring} & \xrightarrow{U} & \mathbb{Group} & \xrightarrow{U} & \mathbb{Monoid} & \xrightarrow{U} & \mathbb{SemiGp} \\
 & & & & \Big\uparrow & & & & \Big\uparrow \\
 & & & & \mathbb{AbGp} & & & & \mathbb{Magma}
\end{array}$$

We also highlight these different relationships in this Venn diagram. Inclusion relationships are described with regular lines, while forgetful functors are described with zig-zag lines.

- Just as functions can be composed, so too functors can be composed.
- For $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{C}$, the composition $G \circ F\colon \mathbb{A} \longrightarrow \mathbb{C}$ is defined as $(G \circ F)(a) = G(F(a))$.
- The composition is defined for morphism $f\colon a \longrightarrow a'$ as $(G \circ F)(f) = G(F(f))\colon G(F(a)) \longrightarrow G(F(a'))$.
- If there are two composable morphisms f and f' in $\mathbb{A}$, then we will have expressions like $(G \circ F)(f' \circ f)$.
- The $\circ$ on the left denotes composition of functors and the $\circ$ on the right denotes composition in $\mathbb{A}$.

Examples

Theorem

The composition of two functors is a functor.

Proof.

Given $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{C}$ we have to show that $G \circ F$ is a functor.

- On object a in $\mathbb{A}$, $(G \circ F)(a) = G(F(a))$.
- For $f\colon a \longrightarrow a'$ and $f'\colon a' \longrightarrow a''$ we have

$$(G \circ F)(f' \circ f) = G(F(f' \circ f)) = G(F(f')) \circ G(F(f))$$

$$= (G \circ F)(f') \circ (G \circ F)(f).$$

- For a in $\mathbb{A}$,

$$(G \circ F)(id_a) = G(F(id_a)) = G(id_{F(a)}) = id_{G(F(id_a))} = id_{(G \circ F)(id_a)}$$

□

We are ready to move up a level. Till now we have dealt with two structures and a functor between them. Now we will talk about *all* structures and *all* functors between them.

A Category Defined

The composition of functors and the identity functors bring to light one of the most important examples of a category:

- $\mathbb{CAT}$ *is the category of all categories and functors between them.*
- *We are mostly interested in a subcategory* $\mathbb{C}\mathrm{at}$ *which consists of all locally small categories and functors between them.*

Exercise

Show that $\mathbf{0}$ *is the initial object in* $\mathbb{C}\mathrm{at}$ *and* $\mathbf{1}$ *is a terminal object in* $\mathbb{C}\mathrm{at}$.

Examples

Example

In this text, we have already seen many functors that take values in $\mathbb{Cat}$. When we wrote about them, we did not have the language to describe them as functors. Now we do.

- *Every set can be thought of as a discrete category. For every set S, there is a discrete category $d(S)$. This is a functor $d\colon \mathbb{Set} \longrightarrow \mathbb{Cat}$. The objects of $d(S)$ are the elements of S and the only morphisms are identity morphisms. If $f\colon S \longrightarrow S'$, then there is a corresponding functor $d(f)\colon d(S) \longrightarrow d(S')$ where $d(f)(s) = f(s)$*

Example (Continued.)

- *Every monoid is a one-object category. For every monoid M, there is a one-object category $A(M)$ whose morphisms are the elements of M and whose composition is the monoid multiplication. This is a functor* $A\colon \mathbb{Monoid} \longrightarrow \mathbb{Cat}$. *If* $f\colon M \longrightarrow M'$ *is a monoid homomorphism, then* $f(m \star m') = f(m) \star' f(m')$. *This requirement shows that the obvious map* $A(f)\colon A(M) \longrightarrow A(M')$ *that takes the single object to the single object and respects the composition is a functor.*

Example (Continued.)

- *Every partial order is a category. For every partial order* $(P, \leq)$*, there is a category* $B(P, \leq)$*, or just* $B(P)$*, whose objects are the objects of* P*, and there is a single morphism from* p *to* p' *in* $B(P)$ *if and only if* $p \leq p'$*. This is a functor* $B\colon \mathbb{PO} \longrightarrow \mathbb{Cat}$*. An order preserving* $f\colon (P, \leq) \longrightarrow (P', \leq')$ *induces a functor* $B(f)\colon B(P, \leq) \longrightarrow B(P', \leq')$*. Whatever we said about partial orders is also true for preorders. This means that there is a functor* $\mathbb{PreO} \longrightarrow \mathbb{Cat}$*.*

Example (Continued.)

- *Every group is a one-object category where all the morphisms are isomorphisms. Every group can be thought of as a one-object category where the elements of the group become invertible morphisms in the category. The composition in the category is the group multiplication. The inverse of the morphism is the inverse of the related element. This describes a functor* $C\colon \mathbb{Group} \longrightarrow \mathbb{Cat}$. *Group homomorphisms* $f\colon G \longrightarrow G'$ *become functors* $C(f)\colon C(G) \longrightarrow C(G')$.

Example (Continued.)

- *The powerset of a set is a partial order category. For every set* S*,* $\mathcal{P}(S)$ *has the structure of a partial order. This gives us a functor* $\mathbb{Set} \longrightarrow \mathbb{PO}$*, and we saw that there is a functor* $B\colon \mathbb{PO} \longrightarrow \mathbb{Cat}$*. Composing these two functors gives us* $\mathcal{P}\colon \mathbb{Set} \longrightarrow \mathbb{Cat}$*. Every set function* $f\colon S \longrightarrow S'$ *induces the direct image functor* $\mathcal{P}(f)\colon \mathcal{P}(S) \longrightarrow \mathcal{P}(S')$ *that we met here.*
- *The opposite operation takes a category to its opposite category. For a category* $\mathbb{A}$ *there is a category* $\mathbb{A}^{op}$*. This a functor* $(\ )^{op}\colon \mathbb{Cat} \longrightarrow \mathbb{Cat}$*. It also takes a functor* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *to the functor* $F^{op}\colon \mathbb{A}^{op} \longrightarrow \mathbb{B}^{op}$ *(which is defined in the obvious way.)*

Exercise

Show that $(\ \ )^{op} \circ (\ \ )^{op} = Id_{\mathbb{Cat}}$*, i.e., for any category* $\mathbb{A}$*,* $((\mathbb{A}^{op})^{op}) = \mathbb{A}$*.*

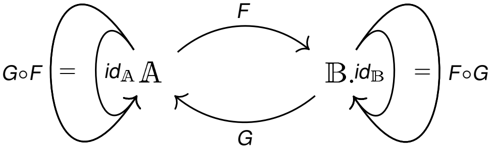
When dealing with sets, we have a method of saying that two sets are essentially the same. We called a set functions $f\colon S \longrightarrow T$ an isomorphism if there exists a $g\colon T \longrightarrow S$ such that for all s in S, $g(f(s)) = s$ and for all t in T, $f(g(t)) = t$. Another way to say that is $g \circ f = Id_S$ and $f \circ g = Id_T$. Set S is isomorphic to set T if there is an ismorphism between them and we write it as $S \cong T$.

We extend this idea to categories.

Definition

Let $\mathbb{A}$ and $\mathbb{B}$ be categories. A functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is an **isomorphism** *if there exists a functor $G\colon \mathbb{B} \longrightarrow \mathbb{A}$ called the* **inverse** *of F such that $G \circ F = Id_{\mathbb{A}}$ and $Id_{\mathbb{B}} = F \circ G$ (we will soon explain why we are writing it this way as opposed to the equivalent $F \circ G = Id_{\mathbb{B}}$.)*

Similar to Diagram (??), we can express this as



Definition

Category $\mathbb{A}$ is **isomorphic** *to category $\mathbb{B}$ if there is an isomorphism between them, and which we write as $\mathbb{A} \cong \mathbb{B}$.*

Exercise

Show that categories $\mathbb{A}$ and $\mathbb{B}$ are isomorphic if and only if there is a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ that is bijective on objects, full, and faithful.

Essentially, an isomorphism of categories means that the categories have exactly the same structure. The two functors essentially rename the objects and morphisms of the categories.

Examples

Example

Let us examine some simple examples of isomorphism of categories.

- *An identity functor is an isomorphism.*
- *We saw that for any category $\mathbb{A}$, the product $\mathbb{A} \times \mathbf{1}$ is essentially the same as $\mathbb{A}$. Specifically, the projection functor $\pi_{\mathbb{A}} \colon \mathbb{A} \times \mathbf{1} \longrightarrow A$ is an isomorphism.*
- *$\mathbb{Rel}^{op} \cong \mathbb{Rel}$. The isomorphism functor takes the set S to S and the relation $R \subseteq S \times T$ to the inverse relation $R^{-1} \subseteq T \times S$. Notice that this works for $\mathbb{Rel}$. It does not work with $\mathbb{Set}$ or $\mathbb{Par}$.*
- *$\mathbf{K}\mathbb{Mat}^{op} \cong \mathbf{K}\mathbb{Mat}$. The isomorphism functor $(\ )^T \colon \mathbf{K}\mathbb{Mat}^{op} \longrightarrow \mathbf{K}\mathbb{Mat}$ is the identity on the natural numbers and takes $A \colon m \longrightarrow n$ to $A^T \colon n \longrightarrow m$. The inverse of the isomorphism functor is itself, i.e., $(A^T)^T = A$.*

Central Idea

Important Categorical Idea

Weakening Structures.

- *The notion of an isomorphism of category is a legitimate idea but it is a very strong requirement.*
- *The necessity that $G \circ F$ is equal to $Id_{\mathbb{A}}$ and $F \circ G$ is equal to $Id_{\mathbb{B}}$ ensures that there are not a lot of interesting, nontrivial examples of isomorphism of categories.*
- *The main problem is that insisting on equality is too strong.*
- *In general, the weaker the assumption, the more phenomena we can encapsulate.*
- *How weak can we go?*
- *This topic is central in higher category theory which we will meet.*

Example

Let $\mathbb{A}$ be a locally small category and $a \in \mathbb{A}$. We know that for every $b \in \mathbb{A}$, there is a set $Hom_{\mathbb{A}}(a, b)$. This brings to light the functor

$$Hom_{\mathbb{A}}(a, \ ) \colon \mathbb{A} \longrightarrow \mathbb{Set}.$$

For every object $b \in \mathbb{A}$ there is a set $Hom_{\mathbb{A}}(a, b)$ and for every $f \colon b \longrightarrow b'$ in $\mathbb{A}$ there is a morphism of sets

$$Hom_{\mathbb{A}}(a, f) \colon Hom_{\mathbb{A}}(a, b) \longrightarrow Hom_{\mathbb{A}}(a, b')$$

which takes $g \colon a \longrightarrow b$ to $f \circ g \colon a \longrightarrow b \longrightarrow b'$ in $Hom_{\mathbb{A}}(a, b')$. That is, $Hom_{\mathbb{A}}(a, f)(g) = f \circ g$. Since this functor is induced by f, we sometimes denote $Hom_{\mathbb{A}}(a, f)$ as f_*.

Example (Continued.)

Let us show that $Hom_{\mathbb{A}}(a, \ )$ *satisfies the requirement of being a functor.*

- $Hom_{\mathbb{A}}(a, \ )$ *preserves composition: for* $f\colon b \longrightarrow b'$ *and* $f'\colon b' \longrightarrow b''$*, then*

  $$Hom_{\mathbb{A}}(a, f' \circ f)(g) = f' \circ f \circ g = (Hom_{\mathbb{A}}(a, f') \circ Hom_{\mathbb{A}}(a, f))(g),$$

  and
- $Hom_{\mathbb{A}}(a, \ )$ *preserves the identity morphisms: for* $id_b\colon b \longrightarrow b$*,*

  $$Hom_{\mathbb{A}}(a, id_b)(g) = id_b \circ g = g = Id_{Hom(a,b)}(g).$$

We will see later that functors of the form $Hom_{\mathbb{A}}(a, \ )$ *are* **representable functors**. *The a represents the functor.*

Types of Functors

- Functors actually come in two flavors: **covariant functors** and **contravariant functors**.
- Although we did not have the name yet, all the functors that we have seen till now are covariant funtors.
- The word “covariant” means the functor varies the same way the source category varies.
- In other words, if there is a morphism from a to a' in the source category, then the covariant functor F takes that morphism from $F(a)$ to $F(a')$ in the target category.
- In stark contrast, a contravariant functor would go “contra” or against (opposite) the way the source category varies.
- In other words, if there is a morphism from a to a' in the source category, then the contravariant functor F takes that morphism from $F(a')$ to $F(a)$ in the target category.

Types of Functors

Definition

Given two categories $\mathbb{A}$ and $\mathbb{B}$, a **contravariant functor** *F from $\mathbb{A}$ to $\mathbb{B}$, written $F\colon \mathbb{A} \longrightarrow \mathbb{B}$, is a rule that assigns to every object a of $\mathbb{A}$ an object $F(a)$ of $\mathbb{B}$ and assigns to every morphism $f : a \longrightarrow a'$ in $\mathbb{A}$, a morphism $F(f)\colon F(a') \longrightarrow F(a)$ in $\mathbb{B}$ (notice the direction). These assignments must satisfy the following two requirements:*

- *Functors reverse the composition of morphisms: for $f\colon a \longrightarrow a'$ and $f'\colon a' \longrightarrow a''$ in $\mathbb{A}$, we require that $F(f' \circ f) = F(f) \circ F(f')$ where the $\circ$ on the left is the composition in $\mathbb{A}$ while the $\circ$ on the right is the composition in $\mathbb{B}$.*
- *Functors respect identity morphisms: for all a in $\mathbb{A}$, we require $F(id_a) = id_{F(a)}$ where id_a is in $\mathbb{A}$ while $id_{F(a)}$ is in $\mathbb{B}$.*

Examples

Example

- *We saw that* $Hom_{\mathbb{A}}(a, \ )$ *is a covariant functor.*
- *In contrast,* $Hom_{\mathbb{A}}(\ , a)$ *is a contravariant functor.*
- *In detail, for every locally small category* $\mathbb{A}$*, and every object* $a \in \mathbb{A}$*, there is a contravariant functor* $Hom_{\mathbb{A}}(\ , a)\colon \mathbb{A} \longrightarrow \mathbb{Set}$.
- *For object* $b \in \mathbb{A}$*, there is a set* $Hom_{\mathbb{A}}(b, a)$*, and for* $f\colon b \longrightarrow b'$ *there is a set function*

$$Hom_{\mathbb{A}}(f, a)\colon Hom_{\mathbb{A}}(b', a) \longrightarrow Hom_{\mathbb{A}}(b, a)$$

  which takes $g\colon b' \longrightarrow a$ *in* $Hom_{\mathbb{A}}(b', a)$ *to* $g \circ f\colon b \longrightarrow b' \longrightarrow a$ *in* $Hom_{\mathbb{A}}(b, a)$.
- *Since this map is induced by* f *and it is contravariant, we sometimes denote* $Hom_{\mathbb{A}}(f, a)$ *as* f^**. Bear in mind the contrast between the covariant* f_* *and the contravvariant* f^*.

Examples

Example (Continued.)

Let us show that $Hom_{\mathbb{A}}(\ \ , a)$ *satisfies the requirement of being a contravariant functor.*

- $Hom_{\mathbb{A}}(\ \ , a)$ *reverses composition: for* $f : b \longrightarrow b'$ *and* $f' : b' \longrightarrow b''$,

  $$Hom_{\mathbb{A}}(f' \circ f, a)(g) = g \circ f' \circ f = (Hom_{\mathbb{A}}(f, a) \circ Hom(f', a))(g),$$

  and
- $Hom_{\mathbb{A}}(\ \ , a)$ *preserves the identity morphisms: for* $id_b : b \longrightarrow b$,

  $$Hom_{\mathbb{A}}(id_b, a)(g) = g \circ id_b = g = id_{Hom(b,a)}(g).$$

We will see later that functors of the form $Hom_{\mathbb{A}}(\ \ , a)$ *are also called* **representable functors**. *The* a *represents the functor.*

Let us apply this to a concept we saw in linear algebra.

Example

We saw that for every set S, the set $Func(S, \mathbf{C})$ has the structure of a complex vector space. If $f\colon S \longrightarrow S'$ is a set function, then $Func(f, \mathbf{C})\colon Func(S', \mathbf{C}) \longrightarrow Func(S, \mathbf{C})$ is a linear map and hence

$$Func(\quad, \mathbf{C}) : \mathbb{Set} \longrightarrow \mathbf{C}\mathbb{Vect}$$

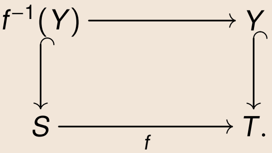
is a contravariant functor.

Example

We saw the covariant direct image functor from $\mathbb{S}\mathrm{et}$ to $\mathbb{S}\mathrm{et}$ that takes set S to $\mathcal{P}(S)$. There is a contravariant version of this. The functor $\mathcal{P}' : \mathbb{S}\mathrm{et} \longrightarrow \mathbb{S}\mathrm{et}$ performs the same action on the objects (i.e., $\mathcal{P}'(S)$ is the powerset of S) but is contravariant. For set function $f\colon S \longrightarrow T$, the functor is defined for $Y \subseteq T$ as

$$\mathcal{P}'(f)(Y) = f^{-1}(Y) = \{x \in S \mid f(x) \in Y\} \subseteq S.$$

This functor can be visualized as



This functor is called the **preimage functor** *or the* **inverse image functor**

There is really no reason to use the nomenclature of contravariant functors for the simple reason that every contravariant functor $F\colon \mathbb{A} \longrightarrow B$ has a related covariant functor $F'\colon \mathbb{A}^{op} \longrightarrow \mathbb{B}$ that performs the same action. Remember that in $\mathbb{A}^{op}$, the arrows are all turned around, and the composition is reversed (see.) Thus the contravariant functors in the previous two Examples can be written as covariant functors

$$Func(\ \ , \mathbf{C}) : \mathbb{Set}^{op} \longrightarrow \mathbf{C}\mathbb{Vect} \qquad \text{and} \qquad \mathcal{P}' : \mathbb{Set}^{op} \longrightarrow \mathbb{Set}.$$

Type of Functor

Functors might have more than one input.

Definition

Given categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, a **bifunctor** $F\colon \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{C}$ *is simply a functor from the product of $\mathbb{A}$ and $\mathbb{B}$ to $\mathbb{C}$. In particular, F is a rule that assigns to every object a of $\mathbb{A}$ and b of $\mathbb{B}$, an object $F(a, b)$ of $\mathbb{C}$, and assigns to every morphism $f : a \longrightarrow a'$ in $\mathbb{A}$ and morphism $g\colon b \longrightarrow b'$ in $\mathbb{B}$, a morphism $F(f, g)\colon F(a, b) \longrightarrow F(a', b')$ in $\mathbb{C}$.*

Definition (Continue.)

These assignments must satisfy the following two requirements:

- *Functors respect the compositions of morphisms: for* $f\colon a \longrightarrow a'$, $f'\colon a' \longrightarrow a''$, $g\colon b \longrightarrow b'$ *and* $g\colon b' \longrightarrow b''$ *we require that*

$$F(f' \circ f, g' \circ g) = F(f', g') \circ F(f, g)$$

  where the $\circ$ *on the right is the composition in* $\mathbb{C}$.
- *Functors respect identity morphisms: for all* a *in* $\mathbb{A}$ *and* b *in* $\mathbb{B}$, *we require* $F(id_{(a,b)}) = id_{F(a,b)}$ *where* $id_{(a,b)}$ *is in* $\mathbb{A} \times \mathbb{B}$ *while* $id_{F(a,b)}$ *is in* $\mathbb{C}$.

Examples

Technical Point

Many times, rather than writing the name of the bifunctor before the input, like $G(a, b)$ we write the name of the bifunctor as an operation between the input, e.g., $a\Box b$. If we write the bifunctor between the inputs, then

$$F(f' \circ f, g' \circ g) = F(f', g') \circ F(f, g)$$

becomes

$$(f' \circ f)\Box(g' \circ g) = (f'\Box g') \circ (f\Box g).$$

This is another instance of an interchange law that we will see in Important Categorical Idea with morphism composition as one operation and the bifunctor as the other operation.

Examples

Bifunctors can be generalized to a **multifunctor** that takes inputs from a finite number of categories

$$F \colon \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n \longrightarrow \mathbb{B}.$$

The functor might be contravariant in some of the inputs. In such a case, we can look at the opposite category of those inputs and only consider covariant functors. For example, The functor

$$Hom_{\mathbb{A}}(\ \ ,\ \ ) \colon \mathbb{A}^{op} \times \mathbb{A} \longrightarrow \mathbb{S}\mathrm{et}.$$

is a bifunctor where the first input works contravariantly and the second input works covariantly.

Examples

Example

Let $\mathbb{A}$ be any category with binary products. We can think of the binary product as a functor Prod that takes two objects of $\mathbb{A}$ as inputs and outputs their product. That is, there is a functor $Prod\colon \mathbb{A}\times\mathbb{A} \longrightarrow \mathbb{A}$ such that $Prod(a, a') = a \times a'$. In particular, the product of two categories can be described this way. That is, there exists a functor $Prod\colon \mathbb{C}\mathbf{at}\times\mathbb{C}\mathbf{at} \longrightarrow \mathbb{C}\mathbf{at}$.

Technical Point

There is a slight problem that we have to worry about with the above Example. A product of two elements in a category is defined up to a unique isomorphism. In general, there is no unique product (or coproduct) of two elements. So which product should the functor choose? Sometimes we will just assume that the functor chooses one. Sometimes we will simply ignore the question because whatever choice we make will work.

Examples

There are a few functors related to the product of two categories.

Example

For any categories $\mathbb{A}$ *and* $\mathbb{B}$*, there are* **projection functors** $\pi_{\mathbb{A}} \colon \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{A}$ *and* $\pi_{\mathbb{B}} \colon \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{B}$*. The functor* $\pi_{\mathbb{A}}$ *is defined on objects as* $\pi_{\mathbb{A}}(a, b) = a$ *and on morphisms* $\pi_{\mathbb{A}}(f, g) = f$*. The other projection is defined analogously. For any categories* $\mathbb{A}$ *and* $\mathbb{B}$*, there is a* **braid functor** $br_{\mathbb{A},\mathbb{B}} \colon \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{B} \times \mathbb{A}$ *that takes object* (a, b) *to* (b, a) *and morphism* (f, g) *to* (g, f)*. For any category* $\mathbb{A}$*, there is a* **diagonal functor** $\Delta_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathbb{A} \times \mathbb{A}$ *that takes* a *to* (a, a) *and is defined similarly for morphisms.*

Examples

The rest of this section will just be examples of functors. Some of these functors will play prominent roles in the rest of this course.

Examples

Example

*Let $(M, \star, e)$ be a monoid. An M-**set** is a set S with a way that the elements of M "act" or change the elements of S. The "action" is a function $\cdot\colon M \times S \longrightarrow S$ which must satisfy the following two commutative diagrams:*

$$
\begin{array}{ccc}
M \times M \times S & \xrightarrow{\star \times id} & M \times S \\
{\scriptstyle id \times \cdot}\downarrow & & \downarrow{\scriptstyle \cdot} \\
M \times S & \xrightarrow[\cdot]{} & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
\{*\} \times S & \xrightarrow{u \times id} & M \times S \\
 & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \cdot} \\
 & & S.
\end{array}
$$

(Reminder: the map $u\colon \{*\} \longrightarrow M$ is the set function that chooses the unit e of the monoid.)

Example (Continue.)

In detail, this means the following requirements are satisfied.

- *The action must respect the monoid multiplication. This means that if m and m' are elements of M, then the action of $m \star m'$ is the same as first acting with m' and then acting with m. That is, for all m, m' in M, and for all $s \in S$, we have that*

  $$(m \star m') \cdot s = m \cdot (m' \cdot s)$$

  where $\star$ is the monoid multiplication.

- *The action of the identity of the monoid does not make any changes. That is, for e, the identity of the monoid, and for all $s \in S$*

  $$e \cdot s = s.$$

Examples

Example (Continue.)

One can view an M-set as a functor. Think of M as a one-object category $A(M)$*. Then any functor* $F\colon A(M) \longrightarrow \mathbb{S}\mathrm{et}$ *is an M-set. The functor F takes the single object* $*$ *to a set, say* S*. For every* $m \in M$ *which corresponds to a morphism* $m\colon * \longrightarrow *$ *in the one-object category, the image of m under F describes the action of m, i.e.,* $F(m)\colon S \longrightarrow S$*. This is simply an instance of the following isomorphism:*

$$Hom(M, Hom(S,S)) \cong Hom(M \times S, S).$$

Examples

Example

There is a functor $\mathcal{O}\colon \mathbb{Top}^{op} \longrightarrow \mathbb{PO}$. *This functor takes a topological space and outputs the partial order of all the open sets. Letting* (T, τ) *be a topological space, we write* $\mathcal{O}(T)$ *(rather than* $\mathcal{O}((T, \tau))$*) for the partial order of all the open sets. If* U *and* U' *are open sets of* T *then* $U \leq U'$ *if and only if* $U \subseteq U'$. *The reason for* $\mathbb{Top}^{op}$ *and not* $\mathbb{Top}$ *is that* $f\colon T \longrightarrow T'$ *is a continuous map of topological spaces if* f^{-1} *takes open sets of* T' *to open sets of* T. *This translates into the fact that* $\mathcal{O}(f)\colon \mathcal{O}(T') \longrightarrow \mathcal{O}(T)$. *It turns out that many properties of a topological space can be determined by simply examining the partial order of open sets. Sometimes this area is humorously called "Pointless Topology". This is also a beginning of topos theory, which can be found in Chapter 9.*

Examples

Example

- *Let us connect the world of relations and matrices.*
- *Consider the natural number n as the set* $\bar{n} = \{1, 2, \ldots, n\}$.
- *Define the full subcategory Nat*$\mathbb{Rel}$ *of* $\mathbb{Rel}$ *to be a category whose objects are these finite sets of natural numbers.*
- *In detail, the morphisms from* $\bar{m} = \{1, \ldots, m\}$ *to* $\bar{n} = \{1, \ldots, n\}$ *are relations from the first set to the second set.*
- *There is a functor* $F\colon$ *Nat*$\mathbb{Rel} \longrightarrow$ *Bool*$\mathbb{Mat}$ *where Bool*$\mathbb{Mat}$ *is the category of matrices with Boolean entries. Functor F is bijective on objects, i.e.,* $F(\bar{m}) = m$.
- *For* $R \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, *we define the matrix* $F(R)$ *as* $(F(R))[i, j] = 1$ *if and only if* $(j, i) \in R$.

Examples

Example

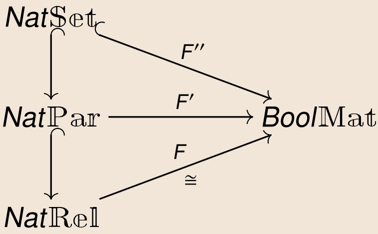
Similar to Nat$\mathbb{R}\mathrm{el}$, there is also Nat$\mathbb{P}\mathrm{ar}$ (objects are the sets of natural numbers and morphisms are partial functions) and Nat$\mathbb{S}\mathrm{et}$ (objects are sets of natural numbers and morphisms are all function.) There are inclusions

$$Nat\mathbb{S}et \hookrightarrow Nat\mathbb{P}ar \hookrightarrow Nat\mathbb{R}el$$

and each of these inclusion are the identity on objects and are faithful but not full.

Example (Continued.)

There are restriction functors to the functor F:



Let us characterize the image of each of these functors.

- *The image of F will be any Boolean matrix.*
- *The image of F' will have any Boolean matrix where there is no more than a single 1 on each row.*
- *The image of F'' will have exactly one 1 on each row.*

Examples

Example

Consider the category $\mathbf{K}\mathbb{M}\mathrm{at}$ *of all matrices with entries in* $\mathbf{K}$*. Let* $\mathbf{K}\mathbb{FDV}\mathrm{ect}$ *be the full subcategory of* $\mathbf{K}\mathbb{V}\mathrm{ect}$ *consisting of all finite dimensional* $\mathbf{K}$ *vector spaces. There is a functor* $F\colon \mathbf{K}\mathbb{M}\mathrm{at} \longrightarrow \mathbf{K}\mathbb{FDV}\mathrm{ect}$ *that takes* m *to the vector space* $\mathbf{K}^m$*. The functor takes the morphism* $A\colon m \longrightarrow n$ *(an* n *by* m *matrix), to the linear transformation* T_A *which is defined for* $B \in \mathbf{K}^m$ *as* $T_A(B) = AB \in \mathbf{K}^n$*. The* F *is a functor because if* $A\colon m \longrightarrow n$ *and* $A'\colon n \longrightarrow p$*, then* $T_A\colon \mathbf{K}^m \longrightarrow \mathbf{K}^n$ *and* $T_{A'}\colon \mathbf{K}^n \longrightarrow \mathbf{K}^p$ *and*

$$F(A' \circ A) = T_{(A' \cdot A)} = T_{A'} \circ T_A = F(A') \circ F(A)$$

and $F(id_m) = T_{Id_{\mathbf{K}^m}} = Id_{\mathbf{K}^m}$.

Examples

Let us consider some examples from logic and computer science.

Example

There is an interesting functor from the category $\mathbb{P}\mathrm{roof}$ of proofs to the category $\mathbb{P}\mathrm{rop}$ of propositions, $Q\colon \mathbb{P}\mathrm{roof} \longrightarrow \mathbb{P}\mathrm{rop}$. The objects of both categories are all propositions of a certain logical system and Q is the identity on objects. The functor Q takes a proof to the implication it proves. In particular, if A and B are two propositions and $f\colon A \longrightarrow B$ is a proof that A implies B (there might be many), then $Q(f)\colon Q(A) \longrightarrow Q(B)$. If Q is full (i.e., every implication has a proof in the system,) then we say that the logical system is **complete**. *In contrast, if Q is not full (i.e., there is an implication that does not have a proof,) then we say that the logical system is* **incomplete**.

Examples

Example (Continued.)

Things get even more interesting when we deal with two logical systems where one system is a subsystem of the other. Then we will have two such functors and a diagram

$$\begin{array}{ccc} \mathbb{P}\mathrm{roof}' & \hookrightarrow & \mathbb{P}\mathrm{roof} \\ {\scriptstyle Q'}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbb{P}\mathrm{rop}' & \hookrightarrow & \mathbb{P}\mathrm{rop}. \end{array}$$

Q' can describe a complete logical system and Q can describe a larger incomplete system. (For example, the right-hand system might be Peano Arithmetic, which we know from Gödel's Incompleteness Theorem, is incomplete. At the same time, the left-hand system is a smaller complete system such as Presburger Arithmetic.).

The next three functors relate matrices, Boolean functions, and logical circuits.

Example

There is a functor FuncDesc $: \mathbb{C}\mathrm{ircuit} \longrightarrow$ *Bool*$\mathbb{F}\mathrm{unc}$ *that describes logical circuits as Boolean functions. Remember that* $\mathbb{C}\mathrm{ircuit}$ *is the category of logical circuits. The objects are the natural numbers and the morphisms are logical circuits. Notice that a circuit with m input wires and n output wires has* 2^m *possible inputs and* 2^n *possible outputs. Consider the category Bool*$\mathbb{F}\mathrm{unc}$ *whose objects are natural numbers and whose morphisms from m to n are the total (computable) functions from* $\{0,1\}^m$ *to* $\{0,1\}^n$*. Composition in Bool*$\mathbb{F}\mathrm{unc}$ *is simple function composition.*

Examples

Example (Continue.)

The functor takes every circuit to the Boolean function it performs. The functoriality means that the composition of circuits gives you the composition of the Boolean functions they perform. This means

$$FuncDesc(C' \circ C) = FuncDesc(C') \circ FuncDesc(C).$$

This functor is the identity on objects and is full. It is not faithful because there are many different logical circuits that perform the same function.

Examples

Example

There is a functor MatrixDesc $: \mathbb{Circuit} \longrightarrow$ *Bool*$\mathbb{Mat}^{op}$ *that describes the operations of a logical circuit with a Boolean matrix. Recall that Bool*$\mathbb{Mat}$ *is the category of Boolean matrices. The objects are the natural numbers and the morphisms are matrices with entries that are either 0 or 1. The functor is defined as follows. On the object m of* $\mathbb{Circuit}$, *MatrixDesc*$(m) = 2^m$. *Circuit C in* $\mathbb{Circuit}$ *with m inputs and n outputs will go to the* 2^m *by* 2^n *matrix MatrixDesc*(C) *whose* i, j *entry is 1 if and only if the jth binary input to C outputs the ith output. We can see how this functor works with some examples of simple logical gates. The inputs are on the top of the matrices and the output is on the left.*

Example (Continued.)

$$MatrixDesc(NOT) = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$MatrixDesc(OR) = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$MatrixDesc(AND) = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$MatrixDesc(NAND) = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Examples

Example (Continued.)

The functor is contravariant because the composition of circuits goes to matrix multiplication in opposite order, i.e., for $C : m \longrightarrow n$ and $C' : n \longrightarrow p$, we have

$$MatrixDesc(C' \circ C) = MatrixDesc(C) \cdot MatrixDesc(C')$$

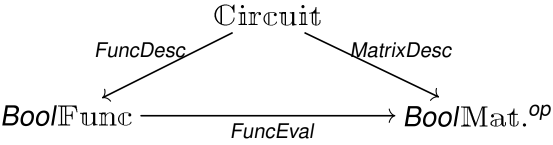
This functor is injective on objects and full, but not faithful. It is full because every such matrix can be described by a circuit. It is not faithful because there can be many different circuits that can describe the same Boolean matrix. A variation of this functor will play a major role in our mini-course on Quantum Computing, Section **??**.

Example

There is a functor FuncEval: *Bool*$\mathbb{Func} \longrightarrow$ *Bool*$\mathbb{Mat}^{op}$ *that describes Boolean functions as Boolean matrices. On objects, FuncEval is like MatrixDesc and takes m to* 2^m. *The functor FuncEval takes a Boolean function* $f\colon \{0,1\}^m \longrightarrow \{0,1\}^n$ *to the* 2^m *by* 2^n *matrix FuncEval*(f) *whose* i, j *entry is 1 if and only if the jth binary input to f outputs the ith output. This functor is contravariant for the same reason as the MatrixDesc is. This functor is injective (but not surjective) on objects, full and faithful.*

Examples

The previous three Examples have to be looked at together. They express a triangle of functors:



It is not hard to see that the triangle commutes.

Examples

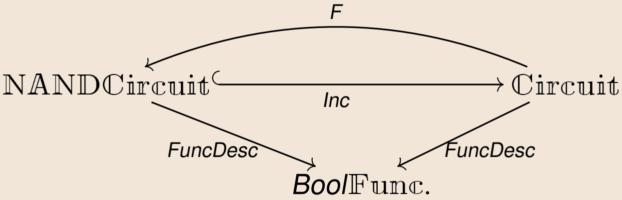
Example

Remember we met the category $\mathbb{NANDCircuit}$. *The objects are the natural numbers, and the set of morphisms from m to n is the set of all logical circuits made of NAND gates that have m input wires and n output wires. This category is a subcategory of* $\mathbb{Circuit}$. *Notice that* $\mathbb{NANDCircuit}$ *and* $\mathbb{Circuit}$ *have the same objects. There is an obvious inclusion functor* $inc\colon \mathbb{NANDCircuit} \hookrightarrow \mathbb{Circuit}$ *that is the identity on the objects.*

Examples

Example (Continued.)

Every circuit in $\mathbb{Circuit}$ is equivalent to a circuit with only NAND gates and fanout operations. This means that there is a functor $F\colon \mathbb{Circuit} \longrightarrow \mathbb{NANDCircuit}$ that is the identity on objects and takes every circuit to an equivalent functor as in the following diagram:



The two triangles commute, and $F \circ Inc = Id$. In general, $Inc \circ F \neq Id$.

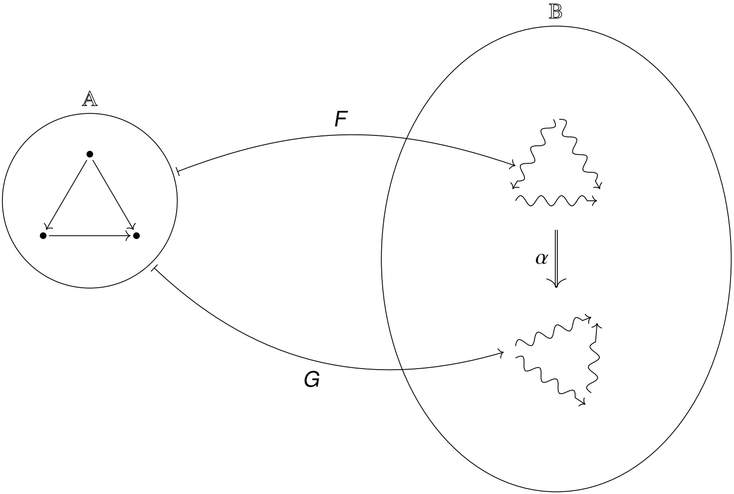
Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.2: Natural Transformations
    - Definitions
    - Isomorphic Natural Transformations
    - Vertical Composition of Natural Transformations
    - Horizontal Composition of Natural Transformations
    - Examples

Definitions

Functors are just the beginning of the story. Just as functors relate categories, so too natural transformations relate functors. A functor goes from a category to a category, while a natural transformation goes from a functor to a functor. Intuitively, if you think of functors $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{B}$ as ways of providing images of $\mathbb{A}$ in $\mathbb{B}$, then a natural transformation from F to G is a way of going from the image of F to the image of G. One can visualize a natural transformation as the diagram on the next slide.

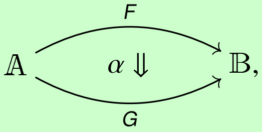
Definitions



Two categories $\mathbb{A}$ and $\mathbb{B}$, two functors *F* and *G*, and a natural transformation α taking the image of *F* to the image of *G*.

Definition

Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *and* $G\colon \mathbb{A} \longrightarrow \mathbb{B}$ *be functors. A* **natural transformation** α *from* F *to* G*, written* $\alpha\colon F \Longrightarrow G$ *or*



is a rule that assigns to every object a *in* $\mathbb{A}$ *a morphism* $\alpha_a\colon F(a) \longrightarrow G(a)$ *called the* **component of** α **at** a.

Definitions

Definition (Continued.)

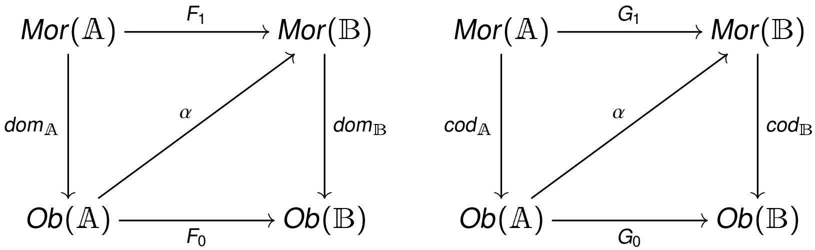
This assignment must further satisfy the following **naturality condition***: for every morphism $f\colon a \longrightarrow a'$ in $\mathbb{A}$, the square*

$$\begin{array}{ccc} F(a) & \xrightarrow{\quad \alpha_a \quad} & G(a) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(a') & \xrightarrow[\quad \alpha_{a'} \quad]{} & G(a') \end{array}$$

commutes.

Notice that a natural transformation is written with a $\Longrightarrow$ but each of its components is written with a $\longrightarrow$.

The functors F and G can be described as we saw. We can add α to those diagrams as in



and insist that the lower triangles commute. Let us explain the squares. The diagonal map takes an object a to $\alpha_a\colon F(a) \longrightarrow G(a)$. The lower triangle in the left square says that α_a goes from $F(a)$, while the lower triangle in the right square says that α_a ends in $G(a)$. Notice that the top triangle do not, in general, commute.

Examples

First some examples:

Example

Just like every category has a unique identity functor, so too, every functor has a unique **identity natural transformation**. *Such a natural transformation does not change the functor. Formally, for every* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *there is a natural transformation* $\iota_F\colon F \Longrightarrow F$ *(*ι *is the Greek letter iota) where each component is* $(\iota_F)_a = id_{F(a)}\colon F(a) \longrightarrow F(a)$.

Example

In computer science there is a **list functor** $List\colon \mathbb{Set} \longrightarrow \mathbb{Set}$ *that takes a set of elements to the set of all lists or sequences of the elements of the set. (this functor was also called the "free monoid functor.") For* $S = \{a, b, c\}$*, we have* $List(S) = \{\emptyset, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \ldots\}$ *For a set map* $f\colon S \longrightarrow T$*, List takes every element in the list to the value of the function. For example, if* $f\colon \{a, b, c\} \longrightarrow \{x, y\}$ *where* $a \mapsto y, b \mapsto x$*, and* $c \mapsto y$*, then the function* $List(f)$ *will take the word* $accbab$ *to the word* $yyyxyx$.

Examples

Example (Continue.)

There are three natural transformations associated to this functor:

- *Reverse*: $List(\ ) \Longrightarrow List(\ )$. *This natural transformation takes a word to the word reversed. For example,* $Reverse_S(accbab) = babcca$.
- *Unit*: $Id_{\mathbb{S}\mathrm{et}}(\ ) \Longrightarrow List(\ )$. *This natural transformation takes an element of the original set (alphabet) to the word of length one that contains that word. For example,* $Unit_S(b) = b$.
- *Flatten*: $List(List(\ )) \Longrightarrow List(\ )$. *This natural transformation takes a list of lists and makes it into a list of words. For example, for the set* $S = \{a, b, c\}$ *an element of* $List(List(S))$ *is* $aba, bbbbc, caaba, bca$. *This element will go to the element* $ababbbbccaababca$.

Example

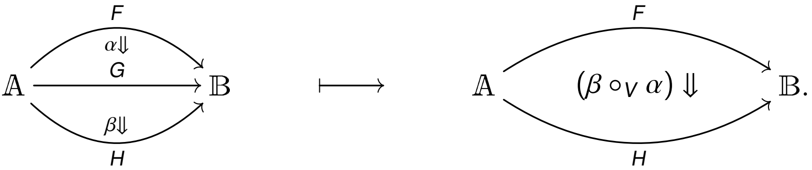
Given two M sets $\cdot\colon M \times S \longrightarrow S$ *and* $\cdot'\colon M \times S' \longrightarrow S'$, *an* ***M*-set homomorphism** *is a set function* $f\colon S \longrightarrow S'$ *that respects the action. This means that for all* $m \in M$ *and* $s \in S$, f *satisfies* $f(m \cdot s) = m \cdot' f(s)$. *In terms of diagrams this means that the following square*

$$\begin{array}{ccc} M \times S & \xrightarrow{id_M \times f} & M \times S' \\ {\scriptstyle \cdot}\downarrow & & \downarrow{\scriptstyle \cdot'} \\ S & \xrightarrow[f]{} & S' \end{array}$$

commutes. If $F\colon M \longrightarrow \mathbb{Set}$ *and* $G\colon M \longrightarrow \mathbb{Set}$ *describe M-sets, then a natural transformation from F to G is the same thing as an M-set homomorphism.*

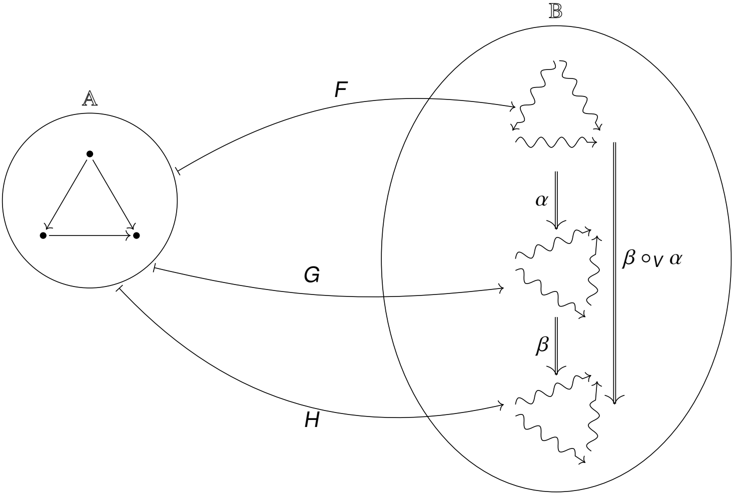
Compositions

There are two types of composition of natural transformations: vertical composition and horizontal composition. **Vertical composition**, $\circ_V$, takes a natural transformation $\alpha\colon F \Longrightarrow G$ and a natural transformation $\beta\colon G \Longrightarrow H$ and gives a natural transformation $\beta \circ_V \alpha\colon F \Longrightarrow H$. This can be visualized as follows:



The next slide has a drawing of how to think of it.

Compositions



Vertical composition of natural transformations.

The component of $\beta \circ_V \alpha$ on element a of $\mathbb{A}$ is defined as

$$(\beta \circ_V \alpha)_a = \beta_a \circ \alpha_a \colon F(a) \longrightarrow G(a) \longrightarrow H(a)$$

which is natural because each of the following squares are commutative and hence the whole diagram is commutative

$$\begin{array}{ccccc}
F(a) & \xrightarrow{\quad \alpha_a \quad} & G(a) & \xrightarrow{\quad \beta_a \quad} & H(a) \\
{\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} & & \big\downarrow{\scriptstyle H(f)} \\
F(a') & \xrightarrow[\quad \alpha_{a'} \quad]{} & G(a') & \xrightarrow[\quad \beta_{a'} \quad]{} & H(a').
\end{array}$$

Remark

The collection of functors from $\mathbb{A}$ *to* $\mathbb{B}$ *and natural transformations between such functors form a category. We call such a structure a* **functor category** *and denote it as* $\mathbb{B}^{\mathbb{A}} = Hom_{\mathbb{Cat}}(\mathbb{A}, \mathbb{B})$*. We will formally meet functor categories soon.*

Compositions

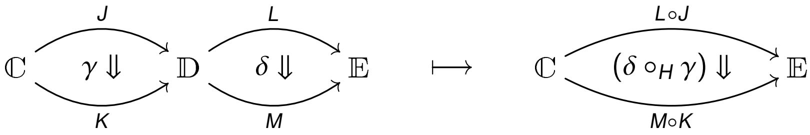
With the notion of vertical composition of natural transformation, we can talk about isomorphic natural transformations.

Definition

A natural transformation $\alpha\colon F \Longrightarrow G$ *is a* **natural isomorphism** *if there exists a* $\beta\colon G \Longrightarrow F$ *such that* $\beta \circ_V \alpha = \iota_F$ *and* $\alpha \circ_V \beta = \iota_G$*. In such a case, F and G are called* **isomorphic functors***.*

It can easily be seen that a natural transformation α is a natural isomorphism if and only if every one of its components $\alpha_a\colon F(a) \longrightarrow G(a)$ is a isomorphism.

Horizontal composition, $\circ_H$, can be visualized as follows in withe the diagram in the next slide.



where the $c \in \mathbb{C}$ component of $\delta \circ_H \gamma$ is defined to be

$$(\delta \circ_H \gamma)_c = \delta_{K(c)} \circ L(\gamma_c) \colon L(J(c)) \longrightarrow L(K(c)) \longrightarrow M(K(c)).$$

This is equivalent to defining it as

$$(\delta \circ_H \gamma)_c = M(\gamma_c) \circ \delta_{J(c)} \colon L(J(c)) \longrightarrow M(J(c)) \longrightarrow M(K(c)).$$

Compositions

These two definitions are equivalent because of the following square in $\mathbb{E}$ commutes out of naturality of δ:

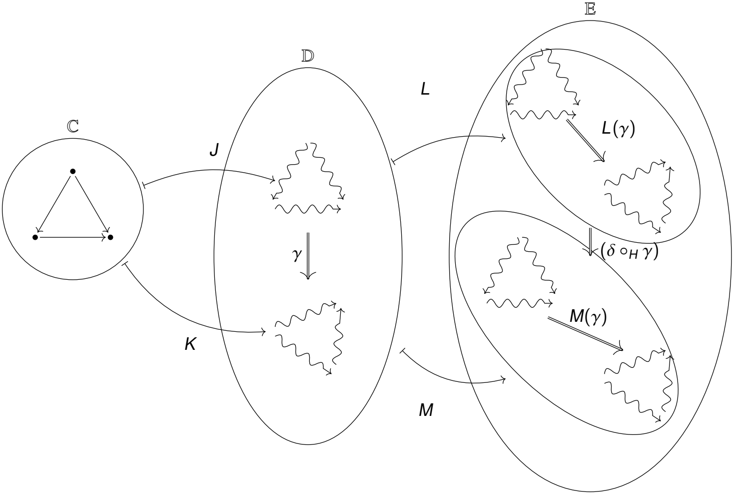
$$\begin{array}{ccc} LJ(c) & \xrightarrow{\delta_{J(c)}} & MJ(c) \\ {\scriptstyle L(\gamma_c)}\downarrow & \searrow^{(\delta \circ_H \gamma)_c} & \downarrow{\scriptstyle M(\gamma_c)} \\ LK(c) & \xrightarrow[\delta_{K(c)}]{} & MK(c). \end{array}$$

The mapping $\delta \circ_H \gamma$ is natural because for all $f\colon c \longrightarrow c'$ in $\mathbb{C}$ there is

$$\begin{array}{ccccc} LJ(c) & \xrightarrow{\delta_{Jc}} & MJ(c) & \xrightarrow{M(\gamma_c)} & MK(c) \\ {\scriptstyle LJ(f)}\downarrow & & {\scriptstyle MJ(f)}\downarrow & & \downarrow{\scriptstyle MK(f)} \\ LJ(c') & \xrightarrow{\delta_{Jc'}} & MJ(c') & \xrightarrow{M(\gamma_{c'})} & MK(c'). \end{array}$$

The left square commutes because of the naturality of δ. The right square commutes because of the naturality of γ and the functoriality of M.

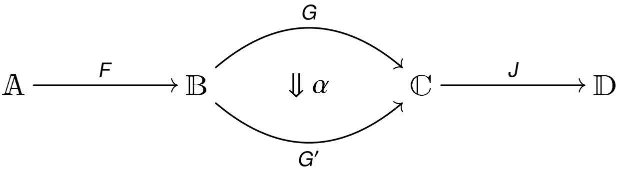
Compositions



Horizontal composition of natural transformations.

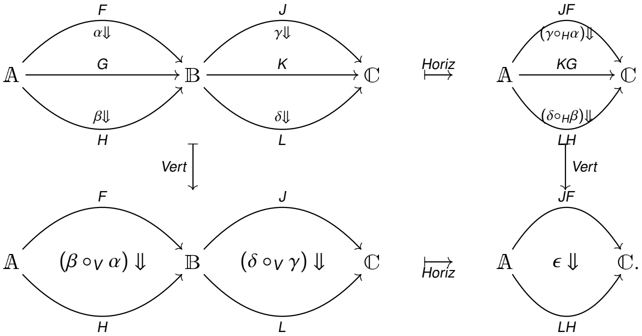
Compositions

When there is a natural transformation of the following form



then we consider the unwritten identity natural transformations ι_F and ι_J. We denote the horizontal composition of $\alpha \circ_H \iota_F$ as αF and $\iota_J \circ_H \alpha$ as $J\alpha$. These compositions of functors and natural transformations are sometimes called **whiskering**.

The vertical and horizontal compositions relate as follows:



Compositions

The vertical maps are describing vertical composition of natural transformations while the horizontal maps are horizontal compositions. The diagram shows two ways to go from the four natural transformations to one natural transformation. Using the definitions of the compositions gives the same natural transformation. This means that the $\epsilon \Downarrow$ in the bottom right corner is

$$(\delta \circ_V \gamma) \circ_H (\beta \circ_V \alpha) = (\delta \circ_H \beta) \circ_V (\gamma \circ_H \alpha).$$

The fact that these two compositions form the same natural transformation is another instance of the interchange law, which is another instance of Important Categorical Idea. Here the two operations are vertical and horizontal composition of natural transformations.

Remark

While $\mathbb{CAT}$ *and* $\mathbb{C}\mathrm{at}$ *each contain categories and functors, they are not the complete picture. Natural transformations are a third level of structure. When all three levels are gathered, one has the structure of a* **2-category**. *Such structures contain objects (also called* **0-cells***), morphisms between objects (also called* **1-cells***), and* **2-cells** *between morphisms. We denote a 2-category as a category with a line above it as in,* $\overline{\mathbb{A}}$. *In particular, the 2-category versions of* $\mathbb{CAT}$ *and* $\mathbb{C}\mathrm{at}$ *are denoted* $\overline{\mathbb{CAT}}$ *and* $\overline{\mathbb{C}\mathrm{at}}$*, respectively. We will see more about 2-categories throughout the rest of this text. They will be formally defined and discussed in terms of higher category theory.*

The rest of this section contains examples of natural transformations.

Examples

Example

This example a natural isomorphism from the first few pages of the Eilenberg and Mac Lane's premier paper introducing category theory. It is a motivating example to show the importance of the naturality condition.

Examples

Example (Continue.)

There is a functor $(\ \ )^* \colon \mathbf{K}\mathbb{FDVect} \longrightarrow \mathbf{K}\mathbb{FDVect}^{op}$ *that takes a vector space* V *to its* **dual vector space** $V^* = Hom_{\mathbf{K}\mathbb{FDVect}}(V, \mathbf{K})$*. We met the dual vector space at the end of our mini-course on basic linear algebra. Let* V *be a finite dimensional* $\mathbf{K}$*-vector space with basis* $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$*. Then* V^* *is a set of functions that has the structure of a finite dimensional vector space and has a basis* $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ *where each* $f_i \colon V \longrightarrow \mathbf{K}$ *is defined as* $f_i(e_j) = 1$ *if* $i = j$ *and* $f_i(e_j) = 0$ *otherwise. For an arbitrary* $v \in V$ *where* $v = k_1e_1 + k_2e_2 + \cdots + k_ne_n$*, we have* $f_i(v) = k_i$*.*

Examples

Example (Continue.)

There is a perfectly legitimate isomorphism of vector spaces $\varphi_{V,\mathcal{E},\mathcal{F}}\colon V \longrightarrow V^*$ which depends on $\mathcal{E}$ and $\mathcal{F}$. It is defined as

$$\varphi_{V,\mathcal{E},\mathcal{F}}(k_1 e_1 + k_2 e_2 + \cdots + k_n e_n) = k_1 f_1 + k_2 f_2 + \cdots + k_n f_n.$$

The inverse is obviously

$$\varphi^{-1}_{V,\mathcal{E},\mathcal{F}}(k_1 f_1 + k_2 f_2 + \cdots + k_n f_n) = k_1 e_1 + k_2 e_2 + \cdots + k_n e_n.$$

The fact that these isomorphisms depend on a basis has an "unnatural" feeling to it. When two vector spaces are isomorphic, why should the isomorphism depend on how the elements are expressed? It seems more — dare we say — natural for there to be an isomorphism independent of how the elements are expressed.

Example (Continue.)

There is an isomorphism that does not depend on how it is presented. Compose the functor $(\ )^*\colon \mathbf{K}\mathbb{FDVect} \longrightarrow \mathbf{K}\mathbb{FDVect}^{op}$ *with itself to get a covariant functor* $(\ )^{**}\colon \mathbf{K}\mathbb{FDVect} \longrightarrow (\mathbf{K}\mathbb{FDVect}^{op})^{op} = \mathbf{K}\mathbb{FDVect}$ *which takes every vector space* V *to its "double dual"* V^{**}*. The elements of*

$$V^{**} = Hom_{\mathbf{K}\mathbb{FDVect}}(Hom_{\mathbf{K}\mathbb{FDVect}}(V, \mathbf{K}), \mathbf{K})$$

are functions $\psi\colon\ V^* \longrightarrow \mathbf{K}$*.*

Example (Continue.)

There is an isomorphism of vector spaces $\theta_V\colon V \longrightarrow V^{**}$ *that is defined on an element* $v \in V$ *to be the function that always evaluates on the element* v*. That is,*

$$\theta_V(v) = ev[v]$$

where the function $ev[v]\colon V^* \longrightarrow \mathbf{K}$ *is defined for any linear map* $f\colon V \longrightarrow \mathbf{K}$ *as*

$$ev[v](f) = f(v).$$

This means that $ev[v]$ *simply evaluates at* v*. The linear transformation* θ_V *is an isomorphism because it is injective, and since* $dim(V) = dim(V^*) = dim(V^{**})$*, we know it is an isomorphism.*

Example (Continue.)

These θ_V *are components of a natural transformation*

$$\mathbf{K}\mathbb{FDV}\mathbb{ect} \underset{(\ )^{**}}{\overset{Id}{\Downarrow \theta}} \mathbf{K}\mathbb{FDV}\mathbb{ect}.$$

The fact that it is natural means that for all linear transformations $T\colon V \longrightarrow W$ *we have*

$$\begin{array}{ccc} V & \xrightarrow{\theta_V} & V^{**} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ W & \xrightarrow[\theta_W]{} & W^{**}. \end{array}$$

Example (Continue.)

In words, we have shown that although a vector space, V, is isomorphic with V^*, the isomorphism depends on the basis. In contrast, V is naturally isomorphic to V^{**}. This means that the isomorphism does not depend on the way the elements are described. This naturality is one of the most important ideas in category theory.

Example

Let $\mathbb{A}$ be a category with finite products. For objects a and b, there is a functor $Hom_{\mathbb{A}}(\ \ , a \times b) \colon \mathbb{A} \longrightarrow \mathbb{Set}$ that takes c to $Hom_{\mathbb{A}}(c, a \times b)$. There is another functor $Hom_{\mathbb{A}}(\ \ , a) \times Hom_{\mathbb{A}}(\ \ , b) \colon \mathbb{A} \longrightarrow \mathbb{Set}$ that takes c to the set $Hom_{\mathbb{A}}(c, a) \times Hom_{\mathbb{A}}(c, b)$. The rule that assigns to every pair of maps f and g, the induced map $\langle f, g \rangle$ is a natural isomorphism. Formally

$$\langle\ ,\ \rangle \colon Hom_{\mathbb{A}}(\ \ , a) \times Hom_{\mathbb{A}}(\ \ , b) \Longrightarrow Hom_{\mathbb{A}}(\ \ , a \times b).$$

Example (Continue.)

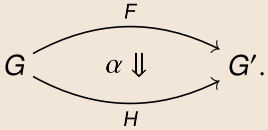
Naturality here means that if there is a function $f\colon c' \longrightarrow c$ in $\mathbb{A}$, then the following diagram commutes:

$$\begin{array}{ccc}
Hom_{\mathbb{A}}(c,a) \times Hom_{\mathbb{A}}(c,b) & \xrightarrow[\cong]{\langle\ ,\ \rangle_c} & Hom_{\mathbb{A}}(c, a \times b) \\
{\scriptstyle f^*\times f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
Hom_{\mathbb{A}}(c',a) \times Hom_{\mathbb{A}}(c',b) & \xrightarrow[\cong]{\langle\ ,\ \rangle_{c'}} & Hom_{\mathbb{A}}(c', a \times b)
\end{array}$$

The fact that this natural transformation is a natural isomorphism follows from our earlier discussion.

Example

The category $\mathbb{G}\mathrm{roup}$ *has groups as objects and homomorphisms between groups as morphisms. There are no nontrivial morphisms between homomorphisms, which means that* $\mathbb{G}\mathrm{roup}$ *does not have 2-cells. However, if we look at groups as one-object categories within the 2-category* $\overline{\mathbb{C}\mathrm{at}}$*, and homomorphisms as functors between such one-object categories, then there are natural transformations between such functors. Let* G *and* G' *be groups thought of as one-object categories and let* $F, H \colon G \longrightarrow G'$ *be functors. Then consider a natural transformation* $\alpha \colon F \Longrightarrow H$ *as in*



Examples

Example (Continue.)

Since there is only one object $*$ in G there is only one component of α, namely $\alpha_* \colon F(*) \longrightarrow H(*)$. The morphism α_* is an element of G'. The naturality condition amounts to commutativity of

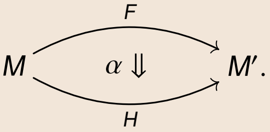
$$\begin{array}{ccc} F(*) & \xrightarrow{\ \alpha_*\ } & H(*) \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle H(g)} \\ F(*) & \xrightarrow[\ \alpha_*\ ]{} & H(*) \end{array}$$

for all morphisms g in G. This means that a natural transformation is an element α_* in G' such that for all g in G, we have that $\alpha_* F(g) = H(g)\alpha_*$ or $F(g) = (\alpha_*)^{-1} H(g)\alpha_*$.

Examples

Example (Continue.)

Notice that, in the same way, we can talk about a natural transformation between two monoid homomorphisms as in



In that case, we have $\alpha_* F(m) = H(m)\alpha_*$*. In general, we cannot write* $F(m) = (\alpha_*)^{-1} H(m)\alpha_*$ *because* α_* *need not be invertible. There, however, might be an invertible element in* M' *and we can insist that our natural transformation uses this invertible element.*

Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.3: Equivalences
    - Definitions
    - Examples

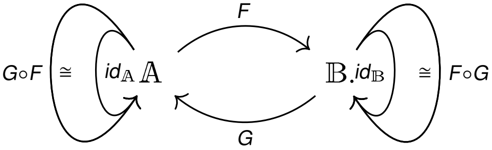
Definitions

- We saw that the notion of isomorphism of categories is very strong and hence does not arise often in a nontrivial way.
- However, if we weaken the notion of isomorphism, we get the notion of equivalence of categories which arises often. (See Important Categorical Idea.)
- If $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ is an isomorphism then for every b in $\mathbb{B}$, there is a unique a in $\mathbb{B}$ such that $F(a) = b$.
- This basically says that the structure of $\mathbb{A}$ and $\mathbb{B}$ are exactly the same.
- Here we weaken this so that for every b in $\mathbb{B}$ there is some a in $\mathbb{A}$ so that $F(a)$ is not necessarily equal to b but is **isomorphic** to b.
- This is the essence of an equivalence of categories.

Definition

Categories $\mathbb{A}$ *and* $\mathbb{B}$ *have an* **equivalence** *between them if there are functors* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *and* $G\colon \mathbb{B} \longrightarrow \mathbb{A}$ *such that* $G \circ F \cong Id_{\mathbb{A}}$ *and* $Id_{\mathbb{B}} \cong F \circ G$*. Functors F and G are called* **quasi-inverses** *of each other and we say that the two categories are* **equivalent categories**. *We denote an equivalence as* $\mathbb{A} \simeq \mathbb{B}$*. Unpacking the definition shows that, for every a in* $\mathbb{A}$*, there is a b in* $\mathbb{B}$*, such that* $G(b)$ *is isomorphic to a, and for every b in* $\mathbb{B}$*, there is an a in* $\mathbb{A}$*, such that* $F(a)$ *is isomorphic to b.*

Similar to this and this about isomorphisms, we can express this as



Let us describe another way of discussing equivalence of categories. From our discussion there arises the following definition of a special type of functor.

Definition

We say a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *is* **essentially surjective**, *if for all* b *in* $\mathbb{B}$, *there is an* a *in* $\mathbb{A}$, *such that* $F(a)$ *is isomorphic to* b.

Theorem

Categories $\mathbb{A}$ and $\mathbb{B}$ are equivalent if and only if there is a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ that is (i) full, (ii) faithful, and (iii) essentially surjective.

Proof.

($\Longleftarrow$) Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be full, faithful, and essentially surjective. We describe G, a quasi-inverse to F. For b in $\mathbb{B}$, we let $G(b)$ be an a, such that $F(a) \cong b$ which definitely exists because F is essentially surjective. (For the cognoscenti, this assumes the axiom of choice.) This will mean that there is an isomorphism $\beta_b\colon FG(b) \longrightarrow b$. Given $h\colon b \longrightarrow b'$ in $\mathbb{B}$, we can form

$$FGb \xrightarrow[\cong]{\beta} b \xrightarrow{h} b' \xrightarrow[\cong]{\beta^{-1}} FGb' \ .$$

Since the source and the target of this morphism is in the image of F and F is full and faithful, there is a unique $\hat{h}\colon Gb \longrightarrow Gb'$ such that $F(\hat{h})$ is this morphism. Set $G(h) = \hat{h}$. This defines G and shows that β is natural. Since $GF(a)$ can equal a and the isomorphism can be the identity, we can see that the $\alpha\colon GF \Longrightarrow Id_{\mathbb{A}}$ is natural. □

Definitions

Proof.

($\Longrightarrow$) Let $\alpha\colon G \circ F \longrightarrow Id_{\mathbb{A}}$ and $\beta\colon F \circ G \longrightarrow Id_{\mathbb{B}}$ be two natural isomorphisms. The functor F is

- Essentially surjective. For every b, there is a $G(b)$ and an isomorphism $\beta_b\colon FG(b) \longrightarrow b$.
- Faithful. If f and f' are morphisms in $\mathbb{A}$ and $F(f) = F(f')$, then by composing with G we have $GF(f) = GF(f')$. By further composing with $\alpha_{a'}$ and α_a^{-1} as in the following diagram

$$\begin{array}{ccc} GFa & \xrightarrow{\alpha_a} & a \\ GFf \downarrow \; = \; \downarrow GFf' & & f \downarrow \;\; \downarrow f' \\ GFa' & \xrightarrow[\alpha_{a'}]{} & a' \end{array}$$

we can see that $\alpha_{a'}(GFf)\alpha_a^{-1} = \alpha_{a'}(GFf')\alpha_a^{-1}$. The naturality of α implies that $f = f'$. Similar arguments show G is faithful.

Proof.

- Full. If $g\colon F(a) \longrightarrow F(a')$, then $G(g)\colon GF(a) \longrightarrow GF(a')$. Let $f\colon a \longrightarrow a'$ be defined as $\alpha_{a'} \circ G(g) \circ \alpha_a^{-1}$. Also consider $GF(f)\colon GF(a) \longrightarrow GF(a')$. We have the commutativity of the following two squares

$$\begin{array}{ccc} GFa & \xrightarrow{\alpha_a} & a \\ {\scriptstyle GFf}\big\downarrow \; \big\downarrow{\scriptstyle Gg} & & \big\downarrow{\scriptstyle f} \\ GFa' & \xrightarrow[\alpha_{a'}]{} & a'. \end{array}$$

One square commutes out of the definition of f and one square commutes out of naturality of α. From this we get $GF(f) = G(g)$. Use the fact that G is faithful and we get that $F(f) = g$. Thus F is full.

□

For a category $\mathbb{A}$, the skeletal category $sk(\mathbb{A})$ includes into $\mathbb{A}$. But we can say more.

Theorem

Every category is equivalent to its skeletal category. Furthermore, if two categories are equivalent, then their skeletal categories are isomorphic, i.e.,

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\ \simeq\ } & \mathbb{B} \\ {\scriptstyle\simeq}\uparrow & & \uparrow{\scriptstyle\simeq} \\ sk(\mathbb{A}) & \xrightarrow[\cong]{} & sk(\mathbb{B}). \end{array}$$

In particular, any two skeletal categories of a category are isomorphic.

Proof.

Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be an equivalence of categories and let a' be an element of $sk(\mathbb{A})$. The functor F takes a' to b in $\mathbb{B}$. There is an element b' in $sk(\mathbb{B})$ that is isomorphic to b. (The axiom of choice was used here.) We make a functor $F'\colon sk(\mathbb{A}) \longrightarrow sk(\mathbb{B})$ that takes a' to b'. For each a there is exactly one such b and vice versa. There is a similar discussion for morphisms. Hence, this functor is an isomorphism. □

Let us look at some examples of equivalence of categories and skeletal categories.

Example

The world's simplest example of an equivalence is the relationship between the one-object category $\mathbf{1}$ *and the category* $\mathbf{2}_I$ *which has two objects and a single isomorphism between them. We can view it as*

$$* \qquad \simeq \qquad a \xrightarrow{\cong} b$$

In detail, there is a unique functor $! : \mathbf{2}_I \longrightarrow \mathbf{1}$ *and there is a functor* $L : \mathbf{1} \longrightarrow \mathbf{2}_I$ *such that* $L(*) = a$*. It is obvious that* $L \circ ! = id_{\mathbf{1}}$ *and that* $! \circ L \cong Id_{\mathbf{2}_I}$*. Another way to say this is that* L *is essentially surjective. Thus we have shown that* $sk(\mathbf{2}_I) = \mathbf{1}$*. Notice that* $\mathbf{2}_I$ *is a groupoid.*

Examples

Example

We can extend this example to all groupoids that have a unique isomorphism between any two objects. Such groupoids are called **contractable**. *They are connected groupoinds where all diagrams commute. It is easy to see that any contractable groupoid is equivalent to* $\mathbf{1}$.

Example

$sk(\mathbb{FinSet}) =$ Nat$\mathbb{Set}$. The skeletal category of $\mathbb{FinSet}$ is Nat$\mathbb{Set}$. The objects of Nat$\mathbb{Set}$ are the sets of natural numbers and the morphisms are all functions between them. There is an inclusion $inc\colon$ Nat$\mathbb{Set} \hookrightarrow \mathbb{FinSet}$ that is full and faithful. The functor inc is essentially surjective because for every finite set S with $|S| = n$, there is an isomorphism between S and the set $\bar{n}$. Thus, it is an equivalence. Since no two elements of Nat$\mathbb{Set}$ are isomorphic, it is a skeletal category.

Examples

Example

- *Every preorder* $(P, \leq)$ *has is a partial order* $(P_0, \leq_0)$ *that is its skeletal category.*
- *There is an inclusion* $inc\colon (P_0, \leq_0) \hookrightarrow (P, \leq)$ *that is full, faithful and essentially surjective (i.e., every* $p \in P$ *is in some isomorphism class represented by an element in* P_0*.)*
- *The quasi-inverse of this map,* $\pi\colon (P, \leq) \longrightarrow (P_0, \leq_0)$*, is also of interest. This function takes every element to its isomorphism representative.*
- *Notice that* $\pi \circ inc = id_{(P_0, \leq_0)}$ *but* $inc \circ \pi$ *need not equal* $id_{(P, \leq)}$.

Examples

Example

Related to the previous example, is the idea that if two partial order categories $(P, \leq)$ *and* $(P', \leq')$ *are equivalent, then they are isomorphic. This is because the only isomorphisms in either category are identities.*

Example

The following equivalence is very useful way of thinking of partial functions. Let $f\colon S \longrightarrow T$ *be a partial function. Now consider the total function* $\hat{f}\colon (S + \{*\}) \longrightarrow (T + \{*\})$ *defined as follows*

$$\hat{f}(x) = \begin{cases} f(x) & : \text{if } f(x) \text{ is defined} \\ * & : \text{if } f(x) \text{ is undefined, or if } x = *. \end{cases}$$

Since $\hat{f}$ *takes the* $*$ *in one set to the* $*$ *in the other set, the function is a map of pointed sets, i.e.,* $*/\mathbb{Set}$*. This describes a functor from* $\mathbb{Par}$ *to* $*/\mathbb{Set}$ *that takes* S *to* $S + \{*\}$*, and takes* f *to* $\hat{f}$*. This functor is full and faithful. The functor going the other way is simpler. Take a function of pointed* $g\colon (S, s) \longrightarrow (T, t)$ *and form the partial function* $g'\colon S \longrightarrow T$ *that is not defined when* $g(x) = t$*. Thus:* $\mathbb{Par} \simeq */\mathbb{Set}$.

Examples

Example

- *There is a functor* $F\colon \mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t} \longrightarrow \mathbf{K}\mathbb{FDV}\mathbb{ect}$ *that is defined on a natural number* m *as* $F(m) = \mathbf{K}^m$.
- *The functor takes the morphism* $A\colon m \longrightarrow n$ *(an* n *by* m *matrix) to the linear transformation* T_A *where* T_A *is defined for* $B \in \mathbf{K}^m$ *is* $T_A(B) = AB$.
- *The functor is full and faithful. The fact that it is essentially surjective follows from the fact that every finite dimensional vector space of dimension* n *is isomorphic to* $\mathbf{K}^n$.
- *Hence we have shown that* $\mathbf{K}\mathbb{M}\mathbb{a}\mathbb{t} \simeq \mathbf{K}\mathbb{FDV}\mathbb{ect}$.
- *The quasi-inverse of* F *takes any* m *dimensional vector space to* m*, and any linear transformation to the matrix that induces it. Notice that the image of the functor* F *is a skeletal category of* $\mathbf{K}\mathbb{FDV}\mathbb{ect}$.

Examples

- Why are equivalences important?
- When two categories are equivalent, they essentially have the same structure and the functors respect that structure.
- This means that if $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is an equivalence, and a is an initial object in $\mathbb{A}$, then $F(a)$ is an initial object of $\mathbb{B}$.
- Even more, if $F(a)$ is the initial object of $\mathbb{B}$, then a is the initial object of $\mathbb{A}$.
- We will see that this is true for most limits and colimits that we met in Chapter 3.

Foreshadowing

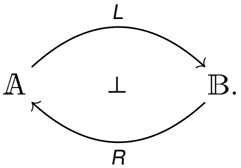
- Chapter 4: Relationships Between Categories
  - Section 4.4: Adjunctions
    - Introduction
    - Definition I
    - Definition II
    - Definition III
    - Definition IV
    - Examples
    - Adjoint Equivalences
    - Composition of Adjunctions

If we weaken the notion of an equivalence of categories, we come to the notion of an adjunction of categories. This weakening of requirements explains why the concept of an adjunction of categories arises very often. Whenever we waken a notion, the new notion will be applicable in more instances (See Important Categorical Idea.) This ubiquity gives adjunctions the status of being one of the most important ideas in all of category theory.

The importance of this notion impels us to give four equivalent definitions of an adjunction between two categories. Let $\mathbb{A}$ and $\mathbb{B}$ be categories and $L\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $R\colon \mathbb{B} \longrightarrow \mathbb{A}$ be functors. If any of the four definitions are satisfied, then all four are satisfied, and we say

- L and R form an **adjunction** between $\mathbb{A}$ and $\mathbb{B}$,
- L is a **left adjoint** of R, and
- R is a **right adjoint** of L.

We denote such an adjunction as $R \vdash L$ or $L \dashv R$ where the dash always points to the left adjoint. We also denote this as

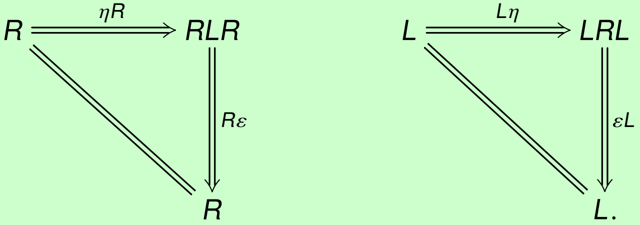


Definition I

The first definition is the easiest way to see an adjunction as a generalization of equivalence.

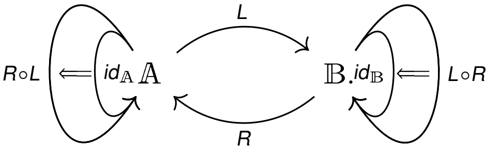
Definition

(I) *There are natural transformations (that need not be natural isomorphisms as in an equivalence of categories)* $\eta\colon Id_{\mathbb{A}} \Longrightarrow R \circ L$ *called the* **unit** *and* $\varepsilon\colon L \circ R \Longrightarrow Id_{\mathbb{B}}$ *called the* **counit**. *The unit and counit must satisfy the following* **triangle identities**



The equal signs mean that the composition of the natural transformations give the original functor.

Similar to this, this, and this, we can express the adjunction as



The second definition stresses the maps between the objects of the two categories.

Definition

(II) *There is an adjunction when there is a certain relationship between the images of the functors that go from category to category. In detail, for all a in* $\mathbb{A}$ *and b in* $\mathbb{B}$ *there is a natural isomorphism of sets*

$$Hom_{\mathbb{B}}(L(a), b) \xrightarrow{\Phi_{a,b}} Hom_{\mathbb{A}}(a, R(b)).$$

Definition II

Definition (Continued.)

Another way to say this is that there is a natural isomorphism between the functors $Hom_{\mathbb{B}}(L(\ \ ),\ \ )$ and $Hom_{\mathbb{A}}(\ \ ,R(\ \ ))$ where both of the functors are of the form $\mathbb{A}^{op} \times \mathbb{B} \longrightarrow \mathbb{Set}$. We can write the fact that these functions are in correspondence with each other as follows

$$
\begin{array}{ccc}
\mathbb{A} & \qquad & a \longrightarrow R(b) \\
L \Big\downarrow \Big\uparrow R & & \\
\mathbb{B} & & L(a) \longrightarrow b.
\end{array}
$$

Because L occurs on the left of the isomorphism, it is called a left adjoint. In contrast, R occurs on the right and is called a right adjoint.

Definition II

Definition (Continued.)

The fact that the isomorphism is natural (or satisfies the naturality condition) means that for every $f\colon a' \longrightarrow a$ *in* $\mathbb{A}$ *and* $g\colon b \longrightarrow b'$ *in* $\mathbb{B}$*, the following two squares commute:*

$$\begin{array}{ccc} Hom_{\mathbb{B}}(L(a),b) & \xrightarrow{\Phi_{a,b}} & Hom_{\mathbb{A}}(a,R(b)) \\ \downarrow{\scriptstyle Hom_{\mathbb{B}}(L(f),b)} & & \downarrow{\scriptstyle Hom_{\mathbb{A}}(f,R(b))} \\ Hom_{\mathbb{B}}(L(a'),b) & \xrightarrow[\Phi_{a',b}]{} & Hom_{\mathbb{A}}(a',R(b)) \end{array}$$

$$\begin{array}{ccc} Hom_{\mathbb{B}}(L(a),b) & \xrightarrow{\Phi_{a,b}} & Hom_{\mathbb{A}}(a,R(b)) \\ \downarrow{\scriptstyle Hom_{\mathbb{B}}(L(a),g)} & & \downarrow{\scriptstyle Hom_{\mathbb{A}}(a,R(g))} \\ Hom_{\mathbb{B}}(L(a),b') & \xrightarrow[\Phi_{a,b'}]{} & Hom_{\mathbb{A}}(a,R(b')). \end{array}$$

Definition (Continued.)

For the sake of clarity, it pays to look at how the maps in the top box work. Remember that the functor L is covariant. However the functor $Hom_{\mathbb{B}}(L(\ ), b)$ *is contravariant, which means it takes* $f\colon a' \longrightarrow a$ *to* $Hom_{\mathbb{B}}(L(f), b)\colon Hom_{\mathbb{B}}(L(a), b) \longrightarrow Hom_{\mathbb{B}}(L(a'), b')$. *In detail, the left vertical set map takes a morphism* $h\colon L(a) \longrightarrow b$ *to*

$$L(a') \xrightarrow{\quad L(f) \quad} L(a) \xrightarrow{\quad h \quad} b.$$

The right set map takes $h'\colon a \longrightarrow R(b)$ *to*

$$a' \xrightarrow{\quad f \quad} a \xrightarrow{\quad h' \quad} R(b).$$

Definition III

The third definition deals with universal property of the round-trip $R \circ L$.

Definition

(III) *There is a universal property that describes the relationship of a starting object to the ending object of this round trip process. Specifically, there exists a natural transformation* $\eta\colon Id_{\mathbb{A}} \Longrightarrow R \circ L$ *called the* **unit**, *which satisfies the following universal property: for any morphism in* $\mathbb{A}$ *of the form* $f\colon a \longrightarrow R(b)$ *there is a unique morphism in* $\mathbb{B}$ *of the form* $f'\colon L(a) \longrightarrow b$ *such that the following triangle commutes:*

$$\begin{array}{ccc} a & \xrightarrow{\eta_a} & R(L(a)) \\ & \searrow^{f} & \downarrow R(f') \\ & & R(b) \end{array} \qquad\qquad \begin{array}{c} L(a) \\ \downarrow f' \\ b. \end{array}$$

Definition IV

The fourth definition deals with universal property of the round-trip $L \circ R$. This definition is very similar to the previous one.

Definition

(IV) *There exists a natural transformation* $\varepsilon\colon L \circ R \Longrightarrow Id_{\mathbb{B}}$ *called the* **counit**, *which satisfies the following universal property: for any morphism in* $\mathbb{B}$ *of the form* $g\colon L(a) \longrightarrow b$, *there is a unique morphism in* $\mathbb{A}$ *of the form* $g'\colon a \longrightarrow R(b)$, *such that the following triangle commutes:*

$$
\begin{array}{ccc}
R(b) & \qquad & L(R(b)) \xrightarrow{\ \varepsilon_b\ } b \\
\uparrow{\scriptstyle g'} & & \uparrow{\scriptstyle L(g')} \quad \nearrow{\scriptstyle g} \\
a & & L(a).
\end{array}
$$

Theorem

All four definitions of adjunctions are equivalent.

The book proves these equivalences very carefully.

Examples

Now for the examples.

Example

We begin with one of the world's simplest example of an adjunction. We generalize what we saw here where we showed that $\mathbf{1} \simeq \mathbf{2}_I$ *to show that* $\mathbf{1}$ *is adjoint to* $\mathbf{2} = \; a \longrightarrow b \;$. *In detail, there is a unique functor* $! : \mathbf{2} \longrightarrow \mathbf{1}$ *and there is a functor* $L : \mathbf{1} \longrightarrow \mathbf{2}$ *such that* $L(*) = a$. *It is obvious that* $L \circ ! = id_{\mathbf{1}}$ *and that*

$$Hom_{\mathbf{1}}(*, !(b)) \cong Hom_{\mathbf{2}}(L(*), b).$$

The set on the right has only one element.

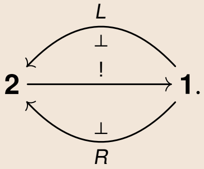
Examples

Example (Continued.)

Similarly, the functor $!$ *has right adjoint* $R\colon \mathbf{1} \longrightarrow \mathbf{2}$ *such that* $R(*) = b$. *This can be seen as*

$$Hom_{\mathbf{1}}(!(b), *) \cong Hom_{\mathbf{2}}(a, R(*)).$$

Both cases are summarized as



Notice that a *is the initial object of* **2**, *while* b *is its terminal object.*

Example

Consider the real numbers $\mathbf{R}$ *and the integers* $\mathbf{Z}$ *as partial order categories. There is an inclusion function* $inc\colon \mathbf{Z} \longrightarrow \mathbf{R}$. *This inclusion function has a left adjoint. Let us call the left adjoint* $L\colon \mathbf{R} \longrightarrow \mathbf{Z}$ *and see if we can figure out what it is. The definition of the adjunction says that for all* n *in* $\mathbf{Z}$ *and for all* r *in* $\mathbf{R}$, *we have*

$$Hom_{\mathbf{Z}}(L(r), n) \cong Hom_{\mathbf{R}}(r, inc(n)).$$

Notice that $inc(n)$ *is just* n *in the real numbers and that both of these categories are partial orders, so the Hom sets are either the empty set or a one-element set. This means that the isomorphism can be interpreted as an if and only if statement:*

$$L(r) \leq n \text{ if and only if } r \leq inc(n) = n.$$

Example (Continued.)

The right hand side is true exactly when r is less than or equal to n. Consider the number 7.27. The following statements are true:

$$7.27 \not\leq 5, \quad 7.27 \not\leq 6, \quad 7.27 \not\leq 7, \quad 7.27 \leq 8, \quad 7.27 \leq 9$$

This forces the left hand side to be true

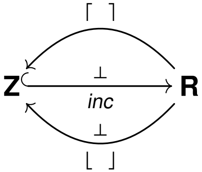
$$L(7.27) \not\leq 5,\ L(7.27) \not\leq 6,\ L(7.27) \not\leq 7,\ L(7.27) \leq 8,\ L(7.27) \leq 9$$

In other words $L(7.27) = 8$. *That is, the left adjoint L is the functor that for an input r, outputs the smallest integer larger than or equal to r. That is the ceiling function. This shows us that* $inc \vdash \lceil\ \rceil$. *The unit* $Id_R \Longrightarrow inc \circ \lceil\ \rceil$ *is the arrow showing that every real number r is less than or equal to its ceiling, i.e.,* $r \leq \lceil r \rceil$. *The counit of this adjunction is the identity. That is, for every natural number n, it is a fact that* $\lceil n \rceil \leq n$.

Exercise

Show that the right adjoint of the inclusion function inc $: \mathbf{Z} \longrightarrow \mathbf{R}$ *is the floor function* $\lfloor \ \rfloor : \mathbf{R} \longrightarrow \mathbf{Z}$. *Describe the unit and counit.*

This past Example and this Exercise can be summarized as



These adjunctions are between partial order categories. Such adjunctions have a special name:

Definition

An adjunction between two preordered or partially ordered categories is called a **Galois connection**. *Definition (II) of an adjunction in the case of preorder or partially ordered categories reduces to*

$$L(a) \leq b \qquad \text{if and only if} \qquad a \leq R(b).$$

Example

We saw that for every set B, there are two functors from $\mathbb{Set}$ to $\mathbb{Set}$: $L_B(A) = A \times B$ and $R_B(C) = Hom_{\mathbb{Set}}(B, C)$. The functor L_B is left adjoint to R_B. Using Definition (II) of adjoint functors amounts to showing that

$$Hom_{\mathbb{Set}}(L_B(A), C) \cong Hom_{\mathbb{Set}}(A, R_B(C))$$

which translates into

$$Hom_{\mathbb{Set}}(A \times B, C) \cong Hom_{\mathbb{Set}}(A, Hom_{\mathbb{Set}}(B, C)).$$

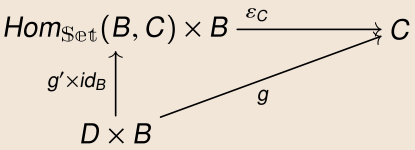
We already saw that these two sets are isomorphic way back in Chapter 1.

Example (Continued.)

The counit of this adjunction will be very important. The counit is a map $\varepsilon\colon L_B \circ R_B \Longrightarrow Id_{\mathbb{Set}}$. *On set* C *this turns out to be*

$$\varepsilon_C\colon Hom_{\mathbb{Set}}(B, C) \times B \longrightarrow C.$$

This morphism takes a set function $f\colon B \longrightarrow C$ *and a* $b \in B$, *and outputs* $f(b)$. *This function is called the* **evaluation function**. *The universal property of the counit says that for every set* D *and every function* $g\colon D \times B \longrightarrow C$, *there is a unique* $g'\colon D \longrightarrow Hom_{\mathbb{Set}}(B, C)$ *such that*



Example (Continued.)

This is true because for a g, we set g' to be the function that takes d to $f_d \colon B \longrightarrow C$, where f_d is defined by $f_d(b) = g(d, b)$. This satisfies the commutative triangle because for any $(d, b) \in D \times B$, we have that $g(d, b) = \varepsilon_C(g'(d), b) = \varepsilon_C(f_d, b)$. This says that the function set, $Hom_{\mathbb{Set}}(A, C)$, is the "best fitting" set to deal with all evaluations.

The unit is a little less familiar and is called the **co-evaluation function.** *The unit is a natural transformation $\eta \colon Id_{\mathbb{Set}} \Longrightarrow R_B \circ L_B$. Its component on set A is $\eta_A \colon A \longrightarrow Hom_{\mathbb{Set}}(B, A \times B)$, which takes $a \in A$ and outputs $f_a \colon B \longrightarrow A \times B$. The function f_a is defined as $b \mapsto (a, b)$. We leave the universal property of the co-evaluation function for the reader.*

The following is a generalization of what we saw.

Example

For every category $\mathbb{A}$ there is a unique functor $! : \mathbb{A} \longrightarrow \mathbf{1}$. The right adjoint, R, of this functor would satisfy the following requirement:

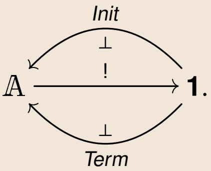
$$Hom_{\mathbb{A}}(a, R(*)) \cong Hom_{\mathbf{1}}(!(a), *) = \{id_*\}.$$

This means that R takes the single object of $\mathbf{1}$ to the object $R(*)$ of $\mathbb{A}$ that has exactly one morphism from any object a of $\mathbb{A}$ to that object. The functor R picks out a terminal object of $\mathbb{A}$ if it exists. The adjoint will exist if and only if the terminal object exists. The unit of the adjunction is the unique map from the object to the terminal object. The counit is always the identity.

Examples

Example (Continued.)

Similarly, the left adjoint of $!$ *picks out an initial object of* $\mathbb{A}$ *if it exists. We can summarize these two adjunctions with*



Example

We saw that any category $\mathbb{A}$ *with products has a functor* $Prod\colon \mathbb{A}\times\mathbb{A} \longrightarrow \mathbb{A}$ *that gives the product of objects and morphisms.saw that for every category* $\mathbb{A}$ *there is a diagonal functor* $\Delta\colon \mathbb{A} \longrightarrow \mathbb{A}\times\mathbb{A}$ *that takes objects and morphisms to their double. These two functors are adjoint to each-other:* $\Delta \dashv Prod$*. In terms of Definition II, this amounts to*

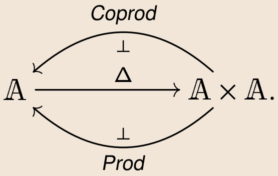
$$Hom_{\mathbb{A}\times\mathbb{A}}(\Delta(c),(a,b)) \cong Hom_{\mathbb{A}}(c, Prod(a,b)).$$

This can be written as

$$Hom_{\mathbb{A}\times\mathbb{A}}((c,c),(a,b)) \cong Hom_{\mathbb{A}}(c, a\times b).$$

Example (Continued.)

There is a similar coprodoct functor coprod : $\mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$. *This functor is left adjoint to* Δ. *We can summarize this as*



The following adjunction is a paradigm for many examples of **free-forgetful adjunctions**.

Example

There is a forgetful functor $U\colon \mathbb{Monoid} \longrightarrow \mathbb{Set}$ *that takes every monoid to its underlying set. We met the free monoid functor* $F\colon \mathbb{Set} \longrightarrow \mathbb{Monoid}$ *that takes every set* S *to* S^**. We show that* F *is left adjoint to* U*. This means that for all sets* S *and for all monoids* M*, there is the following natural isomorphism:*

$$Hom_{\mathbb{Monoid}}(F(S), M) \cong Hom_{\mathbb{Set}}(S, U(M)).$$

Given any set map $f\colon S \longrightarrow U(M)$*, let* $f'\colon F(S) \longrightarrow M$ *be defined as follows: for an element* $w = s_1 s_2 s_3 \cdots s_n$ *of the free monoid,* $f'(w) = f'(s_1 s_2 s_3 \cdots s_n) = f(s_1)f(s_2)f(s_3)\cdots f(s_n)$*. For a monoid homomorphism* $g\colon F(S) \longrightarrow M$*, set the corresponding* $g'\colon S \longrightarrow U(M)$ *to be defined by* $g'(s) = g(s)$*.*

Example (Continued.)

The unit of this adjunction at set S is the set function $\eta_S : S \longrightarrow UF(S)$ which includes every letter as the one-element word. It will be beneficial for us to examine the free monoid on one object, say $*$. The monoid will consist of $*, **, ***, \ldots$. There will also be the empty set as the unit. This monoid is isomorphic to the monoid of natural numbers $(\mathbf{N}, +, 0)$. The universal property of the unit η can be expressed with the diagram

$$\begin{array}{ccc} \{*\} & \xrightarrow{\eta_{\{*\}}} & U(F(\{*\})) = U(\mathbf{N}) \\ & {}_{f}\searrow & \Big\downarrow U(f') \\ & & U(M) \end{array} \qquad\qquad \begin{array}{c} F(\{*\}) = \mathbf{N} \\ \Big\downarrow f' \\ M. \end{array}$$

Example (Continued.)

This says that for every set function $f : \{*\} \longrightarrow U(M)$ *— which is a function that picks out an element* m *of* M *— there is a monoid homomorphism* $f' : \mathbf{N} \longrightarrow M$ *such that the above triangle commutes. The output of the function* f' *is* $m, mm, mmm, \ldots$ *. Let us restate this in a way that will be useful. The free monoid on one object will have the property that for every monoid* M *and every element* m *in* M*, there is a unique morphism from the free monoid on one object to* M *that takes* $*$ *to* m*.*

Examples

Example (Continued.)

Let us summarize the properties of $F(\{*\})$ *in three different ways.*

- *The monoid* $F(\{*\})$ *is the free monoid on one generator.*
- *For every object* m *in* M*, there is a unique morphism* $f\colon F(*) \longrightarrow M$ *such that* $f(*) = m$*.*
- *When we substitute* $S = \{*\}$ *in the main isomorphism here, we get*

$$Hom_{\mathbb{Monoid}}(F(\{*\}), M) \cong Hom_{\mathbb{Set}}(\{*\}, U(M)) \cong U(M).$$

  This means that there is an isomorphism $Hom_{\mathbb{Monoid}}(F(\{*\}), M) \cong U(M)$ *where* $U(M)$ *is the set of elements of* M*.*

When we discuss the free functor of an adjunction we are describing another way of talking about the universal property of a structure. This example will be fundamental in coherence theory.

Examples

The following example will be the paradigm of the next few examples and exercises. The details are worked out for the $\mathbb{C}\mathrm{at}$ example because that is our primary interest.

Example

Consider the categories $\mathbb{C}\mathrm{at}$ *and* $\mathbb{S}\mathrm{et}$. *There is a forgetful functor* $U\colon \mathbb{C}\mathrm{at} \longrightarrow \mathbb{S}\mathrm{et}$ *that takes a category* $\mathbb{A}$ *to the set of objects of* $\mathbb{A}$ *and forgets the morphism and the rest of the structure of the category. It also takes a functor and outputs its underlying set function on objects.*
Functor U *has a left adjoint* $d\colon \mathbb{S}\mathrm{et} \longrightarrow \mathbb{C}\mathrm{at}$ *which takes any set to its "discrete" category and any set function to its "discrete" functor. The name "discrete" is a vestige of topological language.*

Examples

Example (Continued.)

The adjunction is

$$Hom_{\mathbb{Cat}}(d(S), \mathbb{A}) \cong Hom_{\mathbb{Set}}(S, U(\mathbb{A})).$$

The right side of this isomorphism consists of all *set functions from S to $U(\mathbb{A})$. Imagine a set function $f\colon S \longrightarrow U(\mathbb{A})$. This will correspond to the functor $\hat{f}\colon d(S) \longrightarrow \mathbb{A}$. In other words, $\hat{f}$ is the same function as f if we ignore the maps in $\mathbb{A}$.*

The U functor also has a right adjoint, c, for "continuous." This functor corresponds to the "indiscrete" topology. The functor takes a set S and forms the category that has the elements of S as the objects and exactly one morphism between any two objects. The adjunction is

$$Hom_{\mathbb{Cat}}(\mathbb{A}, c(S)) \cong Hom_{\mathbb{Set}}(U(\mathbb{A}), S).$$

Example (Continued.)

A function $f\colon U(\mathbb{A}) \longrightarrow S$ *does not have to worry about the arrows in* $\mathbb{A}$ *because there are no arrows in* $U(\mathbb{A})$. *Such a function will correspond to a functor* $\hat{f}\colon \mathbb{A} \longrightarrow c(S)$ *because for all* a *and* a' *in* $\mathbb{A}$ *there will always be exactly one arrow* $\hat{f}(a) \longrightarrow \hat{f}(a')$ *in* $c(S)$ *where all arrows* $a \longrightarrow a'$ *can go to.*

Example (Continued.)

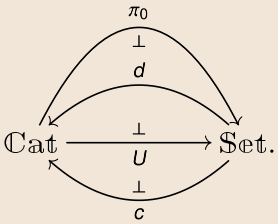
But the story is not over. The left adjoint d *has a left adjoint. There is a functor* π_0 *(again, a vestige of topological language) that takes a category* $\mathbb{A}$ *to* $\pi_0(\mathbb{A})$*, the set of components of* $\mathbb{A}$*. In detail, if there exists a morphism from* a *to* a' *in* $\mathbb{A}$*, then these two elements are in the same component. Let us examine the adjunction:*

$$Hom_{\mathbb{Cat}}(\mathbb{A}, d(S)) \cong Hom_{\mathbb{Set}}(\pi_0(\mathbb{A}), S).$$

Since there are no nonidenity morphisms in the category $d(S)$*, A functor* $F\colon \mathbb{A} \longrightarrow d(S)$ *will take a morphism* $f\colon a \longrightarrow a'$ *in* $\mathbb{A}$ *to an identity morphism in* $d(S)$*. This will correspond to a set function* $\hat{F}\colon \pi_0(\mathbb{A}) \longrightarrow S$*.*

Example (Continued.)

We can summarize all the functors in this example as



Examples

Example (Continued.)

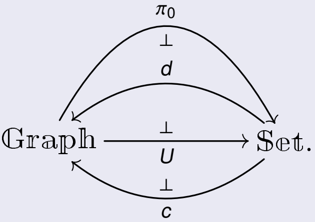
Let us list off the units and counits of these adjunctions.

- *For the $U \vdash d$ adjunction, the unit $\eta_S\colon S \longrightarrow U(d(S))$ takes a set S to the same set. The counit $\varepsilon_{\mathbb{A}}\colon d(U(\mathbb{A})) \longrightarrow \mathbb{A}$ is the inclusion of a category $\mathbb{A}$ stripped of its arrows into the original category $\mathbb{A}$.*
- *For the $c \vdash U$ adjunction, the unit $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow c(U(\mathbb{A}))$ is an identity-on-objects functor from the category $\mathbb{A}$ onto the category with the same objects but with exactly one morphism between any two objects. The counit $\varepsilon_S\colon U(c(S)) \longrightarrow S$ takes a set S to the same set.*
- *For the $d \vdash \pi_0$ adjunction, the unit $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow d(\pi_0(\mathbb{A}))$ takes a category $\mathbb{A}$ to the discrete category of its components. The counit $\varepsilon_S\colon \pi_0(d(S)) \longrightarrow S$ on a set S is the identity on the set, because the components of a discrete category are exactly the same as the elements of the set.*

Examples

Exercise

Graphs are simpler than categories. Use the last example to define the functors and show that the following adjunctions hold between graphs and sets.

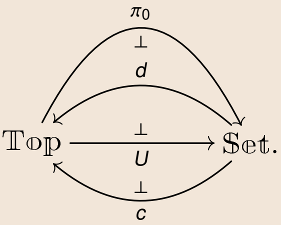


(Hint: The functors and proofs are almost exactly the same as in the previous example.)

Examples

Example

Similar to the above Example and Exercise, there is the following adjunctions between topological spaces and sets.



Examples

Example (Continued.)

The U functor takes a topological space and forgets the topological structure. The output of U is simply the underlying set. The functors d and c take a set to the topological space with the discrete and indiscrete (continuous) topology, respectively (see Example **??***.) The fact that they satisfy the universal properties is exactly the contents of Theorem* **??***. The functor* π_0 *takes a topological spaces and outputs the set of connected components of that space.*

Examples

Example

There is a forgetful functor $U\colon \mathbb{Cat} \longrightarrow \mathbb{Graph}$. *This functor has a left adjoint* $L\colon \mathbb{Graph} \longrightarrow \mathbb{Cat}$. *The functor* L *takes a graph* G *to a category* $L(G)$, *which will be called the "free category over* G*". Such a category will have the same objects as* G *but with more edges added in. In order to make a graph into a category, an identity has to be added in for each object, and a compositions have to be added in for every composable pair of morphisms. In detail,* $L(G)$ *is a category with the same objects as* G *and the morphisms are the set of paths in* G. *Another way to say this is that the morphisms are all composible strings of morphisms in the graph. We might envision them as*

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \cdots \longrightarrow x_n.$$

Example (Continued.)

Included are paths of length zero which correspond to identity morphisms. Composition is simply concatenating two such strings of arrows. Composition with paths of length zero gives the original path, thus insuring that the paths of length zero are the identities. The composition of paths is an associative operation. Hence $L(G)$ *is a category.*

Example (Continued.)

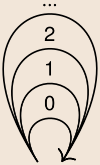
In order to get a feel for this functor, let us examine this functor on two different graphs.

- *The category* $L(*)$ *where* $*$ *is the one-object graph with no arrows. The only morphism added in is the path of length 0, i.e., the identity. This category will be* $\mathbf{1}$*, the one-object category with one identity.*
- *In stark contrast, consider the graph* $*'$ *which consists of one object and one arrow from the single object to itself.*

Examples

Example (Continued.)

Then $L(*')$ *will consist of all possible compositions of that one morphism and will look like*



$*$.

There is one morphism for every natural number. This one-object category is the monoid of natural numbers.
This L *functor is very similar to the free monoid functor from here.*

Examples

Example

We met functors between a partial order and a preorder. These functors can be boosted up to the category of all partial orders and all preorders. There is an inclusion $Inc\colon \mathbb{PO} \hookrightarrow \mathbb{P}\mathrm{re}\mathbb{O}$. *What about going the other way? Let* $(P, \leq)$ *be a preorder. There is a relation* $\approx$ *on the objects of* P *as follows* $p \approx p'$ *if and only if* $p \leq p'$ *and* $p' \leq p$. *This is clearly an equivalence relation. We can form a partial order* $(P/\approx, \sqsubseteq)$ *whose objects are equivalence classes of objects of* P *and* $[p] \sqsubseteq [p']$ *if and only if* $p \leq p'$. *This defines a functor* $\Pi\colon \mathbb{P}\mathrm{re}\mathbb{O} \longrightarrow \mathbb{PO}$. *Notice that* $\Pi \circ Inc = Id_{\mathbb{PO}}$ *but* $Inc \circ \Pi$ *is not equal or isomorphic to* $Id_{\mathbb{P}\mathrm{re}\mathbb{O}}$. *In fact, the map* $Id_{\mathbb{P}\mathrm{re}\mathbb{O}} \Longrightarrow Inc \circ \Pi$ *is the unit of an adjunction with* $Inc \vdash \Pi$. *One can see the adjunction by noticing that any order preserving map of partial orders* $f\colon \Pi(P) \longrightarrow P'$ *has a related order preserving map of preorders* $\widehat{f}\colon P \longrightarrow Inc(P')$, *and vice versa. The map* $\widehat{f}$ *will take isomorphic elements of* P *to the element that* f *took them to.*

Examples

Here is an interesting example about prime numbers.

Example

Consider the functor $L\colon \mathbf{N} \longrightarrow \mathbf{N}$ *which is defined by* $L(n) = P_n$*, the nth prime number. So*

$$L(0) = 0, L(1) = 2, L(2) = 3, L(3) = 5, L(4) = 7, \ldots$$

This functor has a right adjoint $R\colon \mathbf{N} \longrightarrow \mathbf{N}$ *defined as* $R(n) = \pi(n)$*, the number of primes less than or equal to n. So*

$$R(0) = 0, R(1) = 0, R(2) = 1, R(3) = 2, R(4) = 2, \ldots$$

The adjunction says that for all integers m and n,

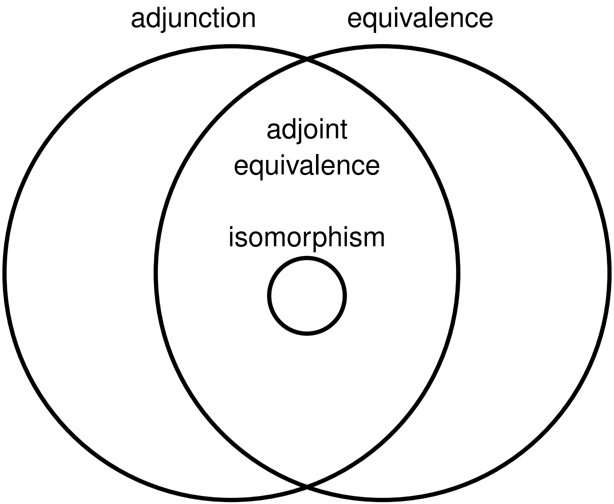
$$P_m \leq n \text{ if and only if } m \leq \pi(n).$$

The unit is actually an equality: $m = \pi(P_m)$*. However, the counit is generally the inequality* $P_{\pi(n)} \leq n$*.*

Properties of Adjunctions

Let us prove some properties about adjunctions. First off, we would like to put the notion of an adjunction in the context of equivalence of categories. We pointed out that an adjunction is a weakening of the notion of an equivalence. That is, every equivalence is a special type of adjunction where the unit and the counit are isomorphisms. But this is not really true. After all, our definition of an equivalence never required the two functors to satisfy the triangle identities. We can rectify the situation by calling an equivalence that also satisfies the triangle identities (and hence all four definitions of an adjunction) an **adjoint equivalence**. This can be visualized as the Venn diagram on the next slide.

Properties of Adjunctions



Adjunctions, equivalences, and adjoint equivalences.

The following theorem relates the functors in the right circle with the functors in the intersection.

Theorem

Every equivalence can be turned into an adjoint equivalence.

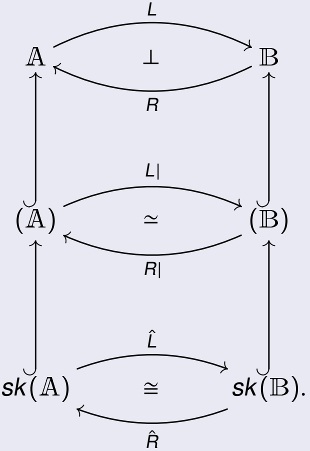
There is another connection between adjunctions and equivalences. Embedded within every adjunction sits an equivalence of categories.

Theorem

For every adjunction $L\colon \mathbb{A} \longrightarrow \mathbb{B}$ *and* $R\colon \mathbb{B} \longrightarrow \mathbb{A}$ *with* $R \vdash L$*, there are subcategories* $(\mathbb{A}) \hookrightarrow \mathbb{A}$ *and* $(\mathbb{B}) \hookrightarrow \mathbb{B}$ *such that* L *and* R *restricted to these subcategories form an equivalence of categories.*

Theorem (Continued.)

We can combine this idea with what we saw here to get the following three layers:



In the top level, the unit and the counit are morphisms, in the middle level, the unit and the counit are isomorphisms, and on the bottom level, the unit and the counit are identity morphisms.

Proof.

Simply let $(\mathbb{A})$ be the full subcategory of A consisting of objects where the unit is an isomorphism, and let $(\mathbb{B})$ be the full subcategory of $\mathbb{B}$ consisting of the objects where the counit is an isomorphism. For the skeletal category level, we have to use a modification of L and R because those functors might not take skeletal objects of one category to skeletal objects of the other category. This can easily be done. □

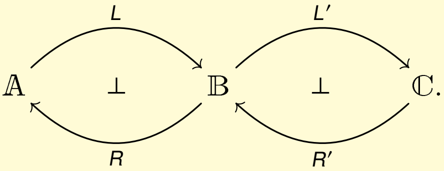
Theorem

The composition of right adjoints is a right adjoint. Similarly the composition of left adjoints is a left adjoint.

Properties of Adjunctions

Proof.

Consider the following two right adjoints.



Then we have

$$Hom_{\mathbb{C}}(R'R(a), c) \cong Hom_{\mathbb{B}}(R(a), L'(c)) \cong Hom_{\mathbb{A}}(a, LL'(c)),$$

where the left natural isomorphism follows from the $R' \vdash L'$ adjunction and the right natural isomorphism follows from the $R \vdash L$ adjunction. The proof for left adjoints is very similar. □

Theorem

A right adjoint to a functor is unique up to a unique isomorphism. That is, if $R \vdash L$ and $R' \vdash L$, then R is isomorphic to R' by a unique isomorphism. There is a similar dual statement about the uniquness of left adjoints.

Properties of Adjunctions

Proof.

From the adjunctions we have

$$Hom(a, R(b)) \cong Hom(L(a), b) \cong Hom(a, R'(b)).$$

Setting $a = R(b)$, this becomes

$$Hom(R(b), R(b)) \cong Hom(L(R(b)), b) \cong Hom(R(b), R'(b)).$$

The first Hom set contains $id_{R(b)}$. The morphism in the third Hom set that corresponds to this is the component τ_b of a natural transformation $\tau\colon R \Longrightarrow R'$. Similarly, setting $a = R'(b)$ in the first line gives

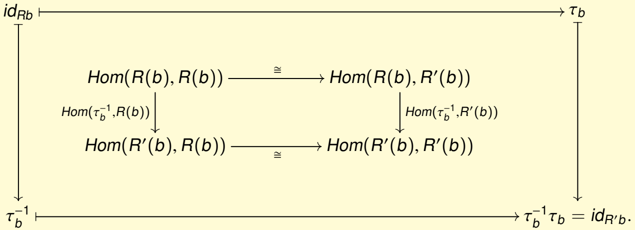
$$Hom(R'(b), R(b)) \cong Hom(LR'(b), b) \cong Hom(R'(b), R'(b)).$$

□

Properties of Adjunctions

Continued.

The third Hom set contains $id_{R'(b)}$. The morphism that corresponds to this in the first Hom set is the b component to τ^{-1}. The naturality of τ and τ^{-1} follows from the naturality of the isomorphisms of the above Hom sets. To see that τ is the inverse of τ^{-1}, consider the following commutative diagram taken from the relevant parts of the above isomorphisms.



There is a similar square to show that $\tau_b\tau_b^{-1} = id_{Rb}$. The uniqueness of τ follows from the fact that these sets are isomorphic and that any such isomorphism has to go to the identity

Properties of Adjunctions

Exercise

Show that if $L: \mathbb{A} \longrightarrow \mathbb{B}$ *is left adjoint to* $R: \mathbb{B} \longrightarrow \mathbb{A}$*, then* $L^{op}: \mathbb{A}^{op} \longrightarrow \mathbb{B}^{op}$ *is right adjoint to* $R^{op}: \mathbb{B}^{op} \longrightarrow \mathbb{A}^{op}$.

$$Hom_{\mathbb{B}^{op}}(b, L^{op}(a)) = Hom_{\mathbb{B}}(L^{op}(a), b) = Hom_{\mathbb{B}}(L(a), b) \cong$$

$$Hom_{\mathbb{A}}(a, R(b)) = Hom_{\mathbb{A}}(a, R^{op}(b)) = Hom_{\mathbb{A}^{op}}(R^{op}(b), a).$$

Properties of Adjunctions

The following definition occurs often.

Definition

Let $\mathbb{A}$ be a full subcategory of $\mathbb{B}$, then $\mathbb{A}$ is called a **reflective subcategory** *if the inclusion has a left adjoint*

$$\mathbb{A} \underset{\perp}{\overset{\longleftarrow}{\hookrightarrow}} \mathbb{B}$$

Dually, a full subcategory is called a **coreflective subcategory** *if the inclusion has a right adjoint.*

Example

We close this section with an important adjunction from linear algebra.

Example

There is a forgetful functor $U\colon \mathbf{K}\mathbb{Vect} \longrightarrow \mathbb{Set}$ *that takes a complex vector space and outputs its underlying set. The functor has a left adjoint* $F\colon \mathbb{Set} \longrightarrow \mathbf{K}\mathbb{Vect}$ *called the* **free vector space functor** *that takes a set and outputs the vector space whose basis is the elements of the set. For example, if the set* $S = \{s_1, s_2, \ldots, s_n\}$ *is a finite set, then the elements of* $F(S)$ *look like this*

$$k_1 s_1 + k_2 s_2 + \cdots + k_n s_n$$

where the k_i *are elements of* $\mathbf{K}$*. The addition of such elements combines like elements. Scalar multiplication is done as*

$$k \cdot (k_1 s_1 + k_2 s_2 + \cdots + k_n s_n) = kk_1 s_1 + kk_2 s_2 + \cdots + kk_n s_n.$$

Example

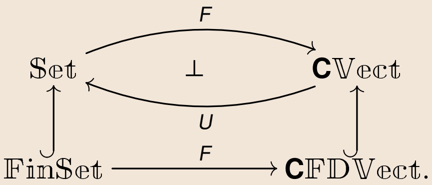
Example (Continued.)

The unit of the adjunction at set S is $\eta\colon S \longrightarrow U(F(S))$, which takes element s to $1s$. This is called "insertion of generators." It is easy to see the universal property of the unit and we leave it to the reader. The functor F is essentially surjective because every vector space is a free vector space. (It is also faithful, but it is not full.) The restriction of F to finite sets outputs finite dimensional vector spaces. Notice that when one forgets the vector space structure of a finite dimensional complex vector space, one does not necessarily get a finite set. This means that there is no forgetful functor from $\mathbf{C}\mathbb{FDV}\mathrm{ect}$ *to* $\mathbb{F}\mathrm{in}\mathbb{S}\mathrm{et}$

Example

Example (Continued.)

This all can all be summarized as



Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.5: Exponentiation and Comma Categories
    - Exponentiation
    - Comma Categories

Foreshadowing

In this section we will meet various operations on categories and functors. We start off discussing functor categories which are the bases of exponentiation. We will also meet comma categories, which are ways of making the morphisms of one category into the objects of another category.

Exponentiation

We have seen that given categories $\mathbb{A}$ and $\mathbb{B}$, we can form category $\mathbb{A} \times \mathbb{B}$. Here we will take two categories and form their functor category. This is analogous to taking two sets, S and T, and forming the set of functions $T^S = Hom_{\mathbb{Set}}(S, T)$.

Definition

Given categories $\mathbb{A}$ and $\mathbb{B}$, there exists the **functor category** *written $\mathbb{B}^{\mathbb{A}}$ or $Hom_{\mathbb{Cat}}(\mathbb{A}, \mathbb{B})$, whose objects are all functors from the category $\mathbb{A}$ to the category $\mathbb{B}$, and whose morphisms are all natural transformations between those functors.*

This fact that for any two objects in $\mathbb{Cat}$, $\mathbb{A}$ and $\mathbb{B}$, the Hom set, $Hom_{\mathbb{Cat}}(\mathbb{A}, \mathbb{B})$, has the structure of a category means that $\mathbb{Cat}$ is a **closed category**.

Let us look at some simple examples.

Example

- *If $\mathbb{A}$ is* **1***, the category with a single object and a single identity morphism, then the objects of $\mathbb{B}^{\mathbf{1}}$ are functors that pick out a single object of $\mathbb{B}$. The natural transformations essentially pick out morphisms of $\mathbb{B}$. We have $\mathbb{B}^{\mathbf{1}} \cong \mathbb{B}$.*
- *If $\mathbb{A}$ is the category $\mathbf{2}_\circ$, the discrete category with two objects and no non-identity morphisms, then a functor $\mathbf{2}_\circ \longrightarrow \mathbb{B}$ picks out two objects of $\mathbb{B}$. The natural transformations are pairs of morphisms in $\mathbb{B}$. This means that*

$$\mathbb{B}^{\mathbf{2}_\circ} \cong \mathbb{B} \times \mathbb{B}.$$

Exponentiation

Example

- *If $\mathbb{A} = \mathbf{2}$, the category with two objects and a morphism between the two objects, then a functor $\mathbf{2} \longrightarrow \mathbb{B}$ picks out a morphism in $\mathbb{B}$. Consider a functor that picks out the morphism $f\colon b_1 \longrightarrow b_2$ and a functor that picks out the morphism $f'\colon b_3 \longrightarrow b_4$, then a natural transformation from the first functor to the second functor amounts to a commutative diagram*

$$\begin{array}{ccc} b_1 & \xrightarrow{\ g\ } & b_3 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ b_2 & \xrightarrow[\ g'\ ]{} & b_4. \end{array}$$

Composition corresponds to horizontal composition of natural transformations and can be seen as pasting one box on top of the other. This functor category is called the ***arrow category***

Example

- *Let $\mathbb{A}$ be the category $* \rightrightarrows *$. Since names of objects and morphisms do not really matter, we might view this category as* $A \underset{trg}{\overset{src}{\rightrightarrows}} V$ *where A stands for arrows, V stands for vertices, src stands for source, and trg stands for target. Then the objects in the functor category* $\mathbb{Set}^{\mathbb{A}}$ *are functors* $F\colon \mathbb{A} \longrightarrow \mathbb{Set}$ *that pick out two sets,* $F(A)$ *and* $F(V)$*, and two set morphisms* $F(src)\colon F(A) \longrightarrow F(V)$ *and* $F(trg)\colon F(A) \longrightarrow F(V)$*. This is nothing more than a directed graph. Natural transformations are exactly directed graph homomorphisms. To summarize,*

$$\mathbb{Set}^{* \rightrightarrows *} \cong \mathbb{Graph}.$$

Example

- *We can take the previous example and go further. Consider the category*

$$\mathbb{A} = \; A \underset{trg}{\overset{src}{\rightrightarrows}} V \longrightarrow L_V.$$

Then the category $\mathbb{Set}^{\mathbb{A}}$ is a directed graph with an added function from the vertices of the graph to a set of labels. This will give the category of graphs with labeled vertices. We also have directed graphs with labeled arrows and labeled vertices:

$$\mathbb{A} = \; L_A \longleftarrow A \underset{trg}{\overset{src}{\rightrightarrows}} V \longrightarrow L_V.$$

Example

- *We already saw that if $\mathbb{A}$ is a monoid, M, which is thought of as a one-object category, then a functor $F\colon M \longrightarrow \mathbb{Set}$ is going to pick out a set S, and for every morphism $m\colon * \longrightarrow *$, there will be a set function $F(m)\colon S \longrightarrow S$. We also saw that natural transformations are homomorphisms of M-sets. Thus $\mathbb{Set}^M$ is the category of M-sets. Similarly, if G is a group, thought of as a one-object category, then $\mathbb{Set}^G$ is the category of G-sets.*

Exponentiation

Example

- *In particular, when the monoid M is the natural numbers* $\mathbf{N}$, *then* $\mathbb{Set}^{\mathbf{N}}$ *is the category of* $\mathbf{N}$*-sets. An object in this category is a set S with maps*

$$S \xrightarrow{F_0} S \xrightarrow{F_1} S \xrightarrow{F_2} \cdots .$$

We can think of these diagrams as describing how systems change in discrete time. The element $s \in S$ *in time t follows the map to become a member of S in time* $t+1$*. The objects in this category are called* **dynamical systems** *or* **discrete time dynamical systems**

Example

- *We can also talk about the monoid* $\mathbf{R}$ *of real numbers. In this case,* $\mathbb{Set}^{\mathbf{R}}$ *becomes* **continuous-time dynamical systems**. *It is harder to draw a diagram of such a system. There is a set* S*, and for every* $r \in \mathbf{R}$ *there is a set function* $t_r \colon S \longrightarrow S$. *Many physical systems can be described by such dynamical systems.*
- *When* $\mathbb{A} = \mathbf{0}$*, the empty category, then for any category* $\mathbb{B}$*, we have* $\mathbb{B}^{\mathbf{0}} = \mathbf{1}$*, because there is exactly one functor from the empty category to any other category.*
- *To what extent can any locally small category* $\mathbb{A}$ *be seen as an functor category? We will see in the Section on the Yoneda Lemma that every small category* $\mathbb{A}$ *can be embedded in a functor category* $\mathbb{Set}^{\mathbb{A}^{op}}$.

Central Idea

Important Categorical Idea

Exponentiation Is Central.

- *Exponentiation is very important throughout category theory.*
- *When we examine $\mathbb{B}^{\mathbb{A}}$, we might think of $\mathbb{A}$ as a "diagram" or a "shape" and think of $\mathbb{B}$ as a context where the diagrams take place.*
- *Other ways to think of $\mathbb{A}$ is as the "ideal model," "syntax," or "cookie cutter" and every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ describes a "semantic model" of the ideal in $\mathbb{B}$ or the "cookie" in $\mathbb{B}$.*

Important Categorical Idea (Continued.)

Exponentiation Is Central.

- *Sometimes we will look at special types of functors* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ *that preserve some of the structures of* $\mathbb{A}$ *and* $\mathbb{B}$ *(e.g. product preserving functors, colimit preserving functors, or monoidal preserving functors, etc.)*
- *Another fundamental idea is the relationship between* $\mathbb{A}$ *(the ideal) and* $\mathbb{B}^{\mathbb{A}}$ *(the collection of models of the ideal.)*
- *It is a particularly important to look at the case when* $\mathbb{B} = \mathbb{S}\mathrm{et}$. *We will examine the relationship between* $\mathbb{A}$ *and* $\mathbb{S}\mathrm{et}^{\mathbb{A}^{op}}$ *when we talk of the Yoneda Lemma.*

Exponentiation

How do functor categories relate to each other? We saw that for any category $\mathbb{A}$, there are covariant and contravariant functors $Hom_{\mathbb{A}}(a, \ )$ and $Hom_{\mathbb{A}}(\ , a)$, respectively. Let us spell out the details for the case when $\mathbb{A} = \mathbb{C}\mathrm{at}$. For any category $\mathbb{C}$, the functor $F \colon \mathbb{B} \longrightarrow \mathbb{B}'$ induces a functor

$$F_* = Hom_{\mathbb{C}\mathrm{at}}(\mathbb{C}, F) \colon \mathbb{B}^{\mathbb{C}} \longrightarrow \mathbb{B}'^{\mathbb{C}}$$

which is defined as

$$H \colon \mathbb{C} \longrightarrow \mathbb{B} \qquad \mapsto \qquad F \circ H \colon \mathbb{C} \longrightarrow \mathbb{B} \longrightarrow \mathbb{B}'.$$

For any category $\mathbb{C}$, the functor $G \colon \mathbb{A} \longrightarrow \mathbb{A}'$, induces a functor

$$G^* = Hom_{\mathbb{C}\mathrm{at}}(G, \mathbb{C}) \colon \mathbb{C}^{\mathbb{A}'} \longrightarrow \mathbb{C}^{\mathbb{A}}$$

which is defined as

$$H \colon \mathbb{A}' \longrightarrow \mathbb{C} \qquad \mapsto \qquad H \circ G \colon \mathbb{A} \longrightarrow \mathbb{A}' \longrightarrow \mathbb{C}.$$

Let us go through some examples of these induced functors.

Example

- *Let* $\mathbb{A} = \mathbf{2}_\circ$*. Consider the functor* $H\colon \mathbf{1} \longrightarrow \mathbf{2}_\circ$ *that takes the single object in* $\mathbf{1}$ *to the first object in* $\mathbf{2}_\circ$*. The functor* H^* *takes a functor* $F\colon \mathbf{2}_\circ \longrightarrow \mathbb{B}$ *to a functor* $F \circ H\colon \mathbf{1} \longrightarrow \mathbf{2}_\circ \longrightarrow \mathbb{B}$*, which outputs the first object that F chose. This is essentially the projection functor* $\pi_1 : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$*. The functor* $H' : \mathbf{1} \longrightarrow \mathbf{2}_\circ$ *that chooses the second object induces the other projection function, i.e.,* $H'^* = \pi_2$*.*

Example

- *Let* $\mathbb{A} = \ A \underset{trg}{\overset{src}{\rightrightarrows}} V.$ *Consider the functor* $J\colon \mathbf{1} \longrightarrow \mathbb{A}$ *that takes the single object in* $\mathbf{1}$ *to the* V *in* $\mathbb{A}$*. Any functor* $F\colon \mathbb{A} \longrightarrow \mathbb{Set}$ *is a graph. The composition with* J *gives a functor* $F \circ J\colon \mathbf{1} \longrightarrow \mathbb{A} \longrightarrow \mathbb{Set}$ *which picks out the set of vertices of the graph. So* J^* *is the forgetful functor from the category* $\mathbb{Graph}$ *to* $\mathbb{Set}$ *that gives the underlying set of vertices. There is a similar functor that gives the underlying set of arrows.*
- *Let* $K\colon \mathbf{1} \longrightarrow M$ *be a functor that takes the single object in the one-object category* $\mathbf{1}$ *to the single object in the one-object category of the monoid* M*. The functor* $K^*\colon \mathbb{Set}^M \longrightarrow \mathbb{Set}^{\mathbf{1}}$ *takes an* M*–set to the underlying set without the action, i.e.,* K^* *forgets the action of the* M*-set.*

Exponentiation

Example

- *Let inc* : $\mathbf{N} \hookrightarrow \mathbf{R}$ *be the inclusion of the one-object monoid of natural numbers to the one-object monoid of real numbers.*
- *The induced functor inc** : $\mathbb{S}\mathrm{et}^{\mathbf{R}} \longrightarrow \mathbb{S}\mathrm{et}^{\mathbf{N}}$ *takes a continuous time dynamical system* $\mathbf{R} \longrightarrow \mathbb{S}\mathrm{et}$ *to the discrete time dynamical system* $\mathbf{N} \longrightarrow \mathbf{R} \longrightarrow \mathbb{S}\mathrm{et}$.
- *The discrete system picks out those parts of the continuous system that correspond to whole numbers.*
- *One way to think of this is that there is some continuous dynamical system, but an experimenter looks at the system at separate time clicks and records the observations.*
- *Much of the physical sciences is done this way.*

Example

There is another example that will be extremely important for the next section and hence worth spending some time spelling out all the ingredients. For every category $\mathbb{A}$, *there is a unique functor* $! \colon \mathbb{A} \longrightarrow \mathbf{1}$. *This functor induces a functor which we denote* Δ

$$\Delta = !^* \colon \mathbb{B}^{\mathbf{1}} \longrightarrow \mathbb{B}^{\mathbb{A}}.$$

In detail, Δ *is defined on objects as*

$$F \colon \mathbf{1} \longrightarrow \mathbb{B} \qquad \mapsto \qquad \Delta(F) = F \circ ! \colon \mathbb{A} \longrightarrow \mathbf{1} \longrightarrow \mathbb{B}.$$

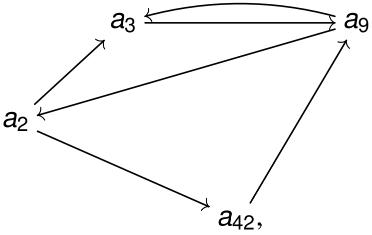
That is, if $F(*) = b$ *(where* $*$ *is the single object in* $\mathbf{1}$*), then* $\Delta(F) = F \circ ! \colon \mathbb{A} \longrightarrow \mathbf{1} \longrightarrow \mathbb{B}$ *is going to take every object of* $\mathbb{A}$ *to* b. *A morphism* $f \colon a \longrightarrow a'$ *in* $\mathbb{A}$ *is going to go to the identity morphism* $id_b \colon b \longrightarrow b$.

Example

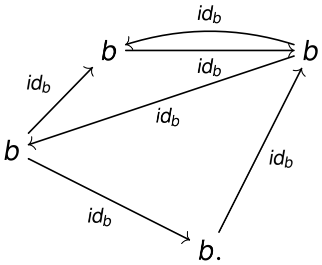
For example, if the category is $\mathbb{A} = \mathbf{2}_\circ$ *then any functor* $F\colon \mathbf{1} \longrightarrow \mathbb{B}$ *that picks out an element* b *will go to the functor* $\Delta(F) = F \circ !\colon \mathbf{2}_\circ \longrightarrow \mathbf{1} \longrightarrow \mathbb{B}$*. This will send each element of* $\mathbf{2}_\circ$ *to* b*. This is exactly the diagonal morphism* $\Delta(b)$*.*

Exponentiation

Now let us generalize from $2_\circ$ to an arbitrary category $\mathbb{A}$. If the category $\mathbb{A}$ is



then one can imagine the image of $\Delta(b)$ as in the next slide.



Of course the image of the functor is simply one object and the identity map, however, we can think of the image as above. Although we are overloading the word, we call Δ the **diagonal functor**. It is similar to the diagonal functor we saw earler where $\Delta\colon \mathbb{A} \longrightarrow \mathbb{A} \times \mathbb{A}$. The old one is defined as $\Delta(a) = (a, a)$. This means that in each position of the output has the value is a. The functor $\Delta\colon \mathbb{B} \longrightarrow \mathbb{B}^{\mathbb{A}}$ also outputs the same value.

Now for constructions that make the morphisms of one category into the objects of another category.

Definition

Given two functors

$$\mathbb{A} \xrightarrow{\quad F \quad} \mathbb{C} \xleftarrow{\quad G \quad} \mathbb{B}$$

we can form the **comma category** (F, G), *sometimes also written* $(F \downarrow G)$. *The objects of this category are triples* (a, f, b) *where* a *is an object of* $\mathbb{A}$, b *is an object* $\mathbb{B}$ *and* $f\colon F(a) \longrightarrow G(b)$ *is a morphism in* $\mathbb{C}$.

Comma Categories

Definition

A morphism from (a, f, b) to (a', f', b') in (F, G) consists of a pair of morphisms (g, h) where $g \colon a \longrightarrow a'$ in $\mathbb{A}$ and $h \colon b \longrightarrow b'$ in $\mathbb{B}$ such that the following square commutes:

$$\begin{array}{ccc} F(a) & \xrightarrow{\quad f \quad} & G(b) \\ {\scriptstyle F(g)}\big\downarrow & & \big\downarrow{\scriptstyle G(h)} \\ F(a') & \xrightarrow[\quad f' \quad]{} & G(b') \end{array} \quad .$$

Composition of morphisms come from the fact that two commuting squares placed one on top of the other, also commute. The identity morphisms are obvious.

Some examples of comma categories are familiar already.

Example

- *If* $\mathbb{A} = \mathbf{1}$ *and* $F\colon \mathbf{1} \longrightarrow \mathbb{C}$ *picks out the object* c_0 *and* $\mathbb{B} = \mathbb{C}$ *with* $G = Id_{\mathbb{C}}$ *as in*

$$\mathbf{1} \xrightarrow{\quad F \quad} \mathbb{C} \xleftarrow{\quad Id_{\mathbb{C}} \quad} \mathbb{C}$$

  then the comma category (F, G) *is the coslice category* $c_0/\mathbb{C}$.

- *If* $\mathbb{B} = \mathbf{1}$ *and* $G\colon \mathbf{1} \longrightarrow \mathbb{C}$ *picks out the object* c_0 *and* $\mathbb{A} = \mathbb{C}$ *with* $F = Id_{\mathbb{C}}$ *as in*

$$\mathbb{C} \xrightarrow{\quad Id_{\mathbb{C}} \quad} \mathbb{C} \xleftarrow{\quad G \quad} \mathbf{1}$$

  then the comma category (F, G) *is the slice category* $\mathbb{C}/c_0$.

Some examples of comma categories are familiar already.

Example

- *If* $\mathbb{A} = \mathbb{B} = \mathbb{C}$ *and* $F = G = Id_{\mathbb{C}}$ *as in*

$$\mathbb{C} \xrightarrow{Id_{\mathbb{C}}} \mathbb{C} \xleftarrow{Id_{\mathbb{C}}} \mathbb{C}$$

then the comma category $(Id_{\mathbb{C}}, Id_{\mathbb{C}})$ *is nothing more than the arrow category* $\mathbb{C}^{\longrightarrow}$ *.*

There are forgetful functors $U_1 : (F, G) \longrightarrow \mathbb{A}$ and $U_2 : (F, G) \longrightarrow \mathbb{B}$ which are defined on objects as follows: functor U_1 takes (a, f, b) to a and U_2 takes (a, f, b) to b. This means that for slice categories $a/\mathbb{A}$ there is a forgetful functor to $\mathbb{A}$. There are similar forgetful functors for coslice categories.

How do comma categories relate with each other? First let us look at slice categories. Consider slice categories $\mathbb{C}/c$ and $\mathbb{C}/c'$. A morphism $g\colon c \longrightarrow c'$ in $\mathbb{C}$ will induce a functor $\mathbb{C}/c \longrightarrow \mathbb{C}/c'$ that takes object $f\colon a \longrightarrow c$ of $\mathbb{C}/c$ to $g \circ f\colon a \longrightarrow c \longrightarrow c'$ in $\mathbb{C}/c'$. Morphisms in $\mathbb{C}/c$ can be dealt with in the same manner. There are similar statements about coslice categories.

Comma Categories

The following theorem shows how general comma categories relate to each other.

Theorem

Given functors

$$\mathbb{A} \xrightarrow{\quad F \quad} \mathbb{C} \xleftarrow{\quad G \quad} \mathbb{B}$$

and

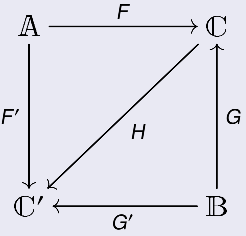
$$\mathbb{A} \xrightarrow{\quad F' \quad} \mathbb{C} \xleftarrow{\quad G' \quad} \mathbb{B},$$

there are comma categories (F, G) *and* (F', G').

Comma Categories

Theorem (Continued.)

Any functor $H\colon \mathbb{C} \longrightarrow \mathbb{C}'$ *where the following two triangles commute*



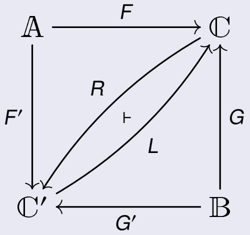
induces a functor $\widehat{H}\colon (F, G) \longrightarrow (F', G')$ *that is defined as*

$$(a, f\colon Fa \longrightarrow Gb, b) \quad \mapsto \quad (a, Hf\colon HFa \longrightarrow HGb, b).$$

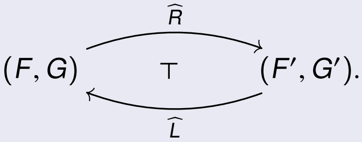
On morphisms, $\widehat{H}$ *takes* (f, g) *to* (f, g)*, i.e., the Hom sets are isomorphic.*

Theorem (Continued.)

Furthermore, if there are adjoint functors $R : \mathbb{C} \longrightarrow \mathbb{C}'$ *and* $L : \mathbb{C}' \longrightarrow \mathbb{C}$ *with* $R \vdash L$ *and the corresponding triangles commute,*



then their induced functors are also adjoint

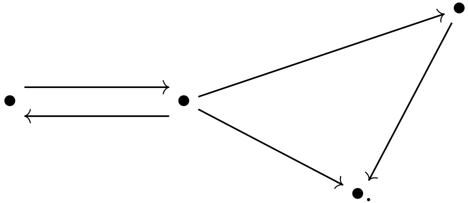


Foreshadowing

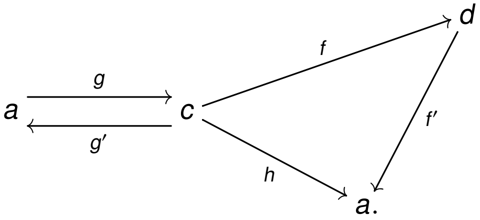
- Chapter 4: Relationships Between Categories
  - Section 4.6: Limits and Colimits Revisited
    - Cones and Cocones
    - Limits and Colimits
    - Preserve and Reflect
    - Completion and Cocompletion

Cones and Cocones

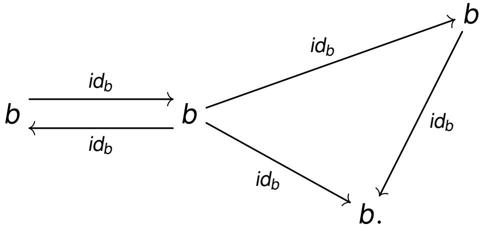
With the knowledge of natural transformations and adjoint functors, we can see limits and colimits in a new light. Let $\mathbb{D}$ be a category which we employ as a diagram in an exponent. We call it a **diagram category** or a **shape category**. For example, category $\mathbb{D}$ might look like



For a category $\mathbb{B}$, the category $\mathbb{B}^{\mathbb{D}}$ is the collection of all functors from $\mathbb{D}$ to $\mathbb{B}$ and natural transformations between them. For example, the image of a typical functor $F\colon \mathbb{D} \longrightarrow \mathbb{B}$ might look like this:

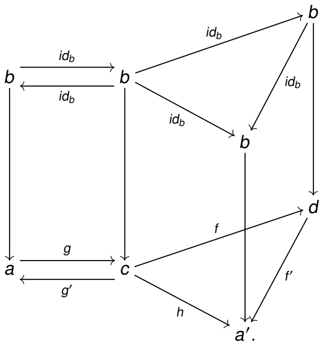


There is a diagonal functor $\Delta\colon \mathbb{B} \longrightarrow \mathbb{B}^{\mathbb{D}}$, which takes every $b \in \mathbb{B}$ to the diagram with all the objects being b and all the morphisms being id_b. For the last diagram, $\Delta(b)$ looks like this:



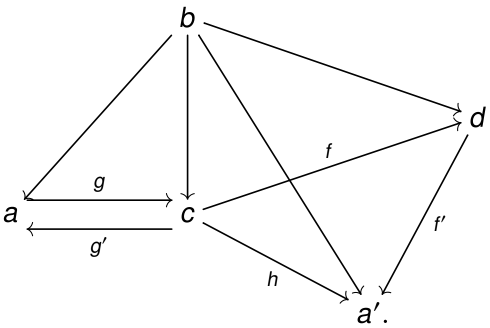
Cones and Cocones

A natural transformation from the functor $\Delta(b)$ to a functor $F\colon \mathbb{D} \longrightarrow \mathbb{B}$ (i.e. a morphism in the category $\mathbb{B}^{\mathbb{D}}$) is a commutative diagram that looks like this



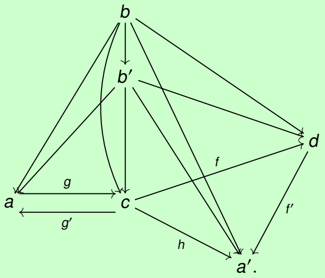
The vertical morphisms are the components of the natural transformation.

We can shorten this to



Definition

A **cone over** $F : \mathbb{D} \longrightarrow \mathbb{B}$ **with base** b *is a natural transformation in* $\mathbb{B}^{\mathbb{D}}$ *from* $\Delta(b)$ *to* F. *We call such a natural transformation. Consider when there are two cones over* F, $\Delta(b) \Longrightarrow F$ *and* $\Delta(b') \Longrightarrow F$. *A map from the cone with base* b *to the cone with base* b' *is a map* $b \longrightarrow b'$ *such that all the expected diagrams commute.*



Definition

With this concept of a morphism between cones over F, we define the category of cones over $F\colon \mathbb{D} \longrightarrow \mathbb{B}$, which we denote $\mathbb{C}\mathrm{one}(F)$.

There is a dual notion of a **cocone over F with base b**, *which is a natural transformation in $\mathbb{B}^{\mathbb{D}}$ from $F\colon \mathbb{D} \longrightarrow \mathbb{B}$ to $\Delta(b)$. There is an obvious definition of a morphism between cocones and the category of cocones over F, denoted $\mathbb{C}\mathrm{ocone}(F)$.*

Cones and Cocones

A formal way of constructing the category of cones for a functor F is to consider the comma category of the following two functors:

$$\mathbb{B} \xrightarrow{\quad\Delta\quad} \mathbb{B}^{\mathbb{D}} \xleftarrow{\quad Const_F\quad} \mathbf{1}$$

where the right functor chooses the functor F. Similarly, the category of cocones of F is the comma category of

$$\mathbf{1} \xrightarrow{\quad Const_F\quad} \mathbb{B}^{\mathbb{D}} \xleftarrow{\quad\Delta\quad} \mathbb{B}\ .$$

With the category of cones, we can talk about the “best fitting” cone.

Definition

The **limit** *of a diagram* $F \colon \mathbb{D} \longrightarrow \mathbb{B}$ *is the terminal object in the category* $\mathbb{Cone}(F)$ *of cones over* F*. That is, it is the cone over* F *with the property that every cone has a unique map to it. In detail, the limit is an object* $Lim(F)$ *of* $\mathbb{B}$ *and a morphism* $Lim(F) \longrightarrow F(d)$ *for every* d *in* $\mathbb{D}$*. The obvious compositions of morphisms commute.*

Definition

The **colimit** *of a diagram* $F\colon \mathbb{D} \longrightarrow \mathbb{B}$ *is the initial object in the category* $\mathbb{Cocone}(F)$ *of cocones over* F*. That is, it is a cocone over* F *with the property that there is a cocone map from it to every cocone over* F*. Again, the colimit is an object* $Colim(F)$ *of* $\mathbb{B}$ *and maps* $F(d) \longrightarrow Colim(F)$ *for every* d *in* $\mathbb{D}$*. The obvious compositions of morphisms commute.*

From the vantage point of looking at limits as terminal objects in the category of cones, one easily sees that a limit has the same uniqueness up to a unique isomorphism as the terminal object of a category.

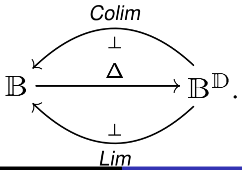
Let us elaborate the details of this definition. A cone is a natural transformation $\Delta(b) \Longrightarrow F$. The limit of F is an object $Lim(F)$ in the category $\mathbb{B}$. A cone of F with the limit as a base of the cone is a natural transform $\Delta(Lim(F)) \Longrightarrow F$. Saying that the limit is the terminal cone means that for every cone $\Delta(b) \Longrightarrow F$ there is a unique morphism $b \longrightarrow Lim(F)$. In terms of Hom sets, this become a statement about adjoint functors:

$$Hom_{\mathbb{B}^{\mathbb{D}}}(\Delta(b), F) \cong Hom_{\mathbb{B}}(b, LimF).$$

There is a similar analysis of coloimits and we get

$$Hom_{\mathbb{B}^{\mathbb{D}}}(F, \Delta(b)) \cong Hom_{\mathbb{B}}(ColimF, b).$$

Both of these adjoint functors can be encapsulated as



Let us examine properties of limits and colimits. First a definition.

Definition

Let $G\colon \mathbb{B} \longrightarrow \mathbb{C}$ be a functor. Notice that for any diagram $F\colon \mathbb{D} \longrightarrow \mathbb{B}$, there is also an induced diagram $G \circ F\colon \mathbb{D} \longrightarrow \mathbb{C}$. We say G **preserves limits** *if for all $F\colon \mathbb{D} \longrightarrow \mathbb{B}$, G takes the limit of F to the limit of $G \circ F$. In symbols, $G(Lim(F)) = Lim(G \circ F)$. We say G* **reflects limits** *if for all $F\colon \mathbb{D} \longrightarrow \mathbb{B}$ and any cone $\lambda\colon \Delta(b) \Longrightarrow F$, the following is true: $G(\lambda)$ is a limit of $G \circ F$ implies λ is a limit of F.*
There are similar definitions about preserving and reflecting colimits.

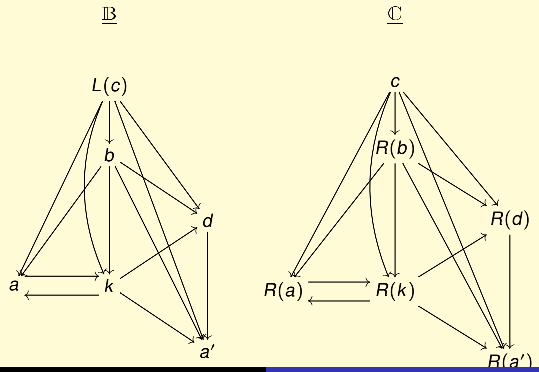
Preserve and Reflect

Theorem

Right adjoints preserve limits and left adjoints preserve colimits.

Proof.

Consider this cone and its image under a right adjoint functor while looking at the proof.



Preserve and Reflect

Continued.

- Let $R\colon \mathbb{B} \longrightarrow \mathbb{C}$ be a right adjoint to $L\colon \mathbb{C} \longrightarrow \mathbb{B}$ and $F\colon \mathbb{D} \longrightarrow \mathbb{B}$.
- The diagram of F is at the bottom left of the figure.
- Assume b is the limit of F in $\mathbb{B}$.
- Apply R to the bottom left to get the diagram on the bottom right.
- Our aim is to show that $R(b)$ is the limit of the diagram on the bottom right.
- By functoriality, there is a map from $R(b)$ to every element of the diagram.

□

Preserve and Reflect

Continued.

- Furthermore, for any c in $\mathbb{C}$ with maps $c \longrightarrow R(x)$ where x is any element of the diagram on the bottom right, there is, by adjointness, a map $L(c) \longrightarrow x$ on the left.
- Since b is the limit on the left, by the universal property of the limit, there is a unique map $L(b) \longrightarrow c$ making all the diagrams commute.
- By adjointness again, there is a unique map $c \longrightarrow R(b)$ on the right ensuring that $R(b)$ is the limit.

□

Remark

It is easy to see that an equivalence of categories is both a right and left adjoint. Such functors preserve and reflect limits and colimits. However there is something very strange with equalizers and coequalizers. Consider an equivalence of categories $F\colon \mathbb{A} \longrightarrow \mathbb{B}$. *Imagine it works as follows:*

$$\underline{\mathbb{A}} \qquad \xrightarrow{F} \qquad \underline{\mathbb{B}}$$

$$a \xrightarrow{f} b \xrightarrow[\cong]{p} c \overset{g}{\underset{h}{\rightrightarrows}} d \qquad\qquad F(a) \xrightarrow{F(f)} \begin{matrix} F(b) \\ = \\ F(c) \end{matrix} \overset{F(g)}{\underset{F(h)}{\rightrightarrows}} F(d)$$

Remark (Continued.)

In the category $\mathbb{A}$ there is an isomorphism $p\colon b \longrightarrow c$. The functor F can have $F(b)$ and $F(c)$ be the same object of $\mathbb{B}$ and $F(p) = id_{F(b)} = id_{F(c)}$. Furthermore, it could be that in the category $\mathbb{B}$, the morphism $F(f) = F(pf)$ is the equalizer of $F(g)$ and $F(h)$. So we have that F is an equivalence of categories, the image $F(f) = F(pf)$ is an equalizer of $F(g)$ and $F(h)$, (pf is an equalizer of g and h), but f is not an equalizer of g and h. (The equivalence still reflects equalizers because in our definition of reflecting limits, we assumed the cone exists in $\mathbb{A}$ already. Here we are talking of just equalizers in the target and not formed in the source.)

Exercise

Show that left adjoints preserve colimits.

Definition

- *A category* **has finite limits** *or is* **finitely complete** *if it has limits of finite diagrams.*
- *A category* **has limits** *or is* **complete** *if it has all limits of small diagrams (the collection of objects and maps are sets).*
- *We similarly define a category that* **has finite colimits** *or is* **finitely cocomplete**, *or is* **cocomplete**.

Theorem

A category is complete if and only if it has all products and equalizers. A category is cocomplete if and only if it has all coproducts and coequalizers.

Central Idea

Important Categorical Idea

The Many Ways of Describing Structures.

- *We have already seen three related ways of describing categorical structures: (i) universal properties, (ii) limits and colimits, and (iii) adjoint functors.*
- *Each of these is important in its own right, and we will continue to talk about each of the three.*
- *However, it is important to realize that they are three ways of talking about the same thing. With each of these, you can describe the other two.*
- *We saw this explicitly, where we learned that limits and colimits are really objects with universal properties (terminal and initial objects) in categories of cones and cocones. These objects are chosen by left and right adjoints.*

Central Idea

Important Categorical Idea (Continued.)

The Many Ways of Describing Structures.

- *We also explicitly saw this in Definition (III) and (IV) of adjoint functors, where the units and counits have universal properties.*
- *It is important to see them individually and as reflections of each other.*
- *We will see that Kan extensions are yet another equivalent way of describing categorical structures.*

Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.7: The Yoneda Lemma
    - Representable Functors
    - Yoneda Embedding Theorem
    - Yoneda Lemma
    - The Contravariant Yoneda Embedding.

As we saw in Important Categorical Idea, the relationship between a category $\mathbb{A}$ and the category $\mathbb{Set}^{\mathbb{A}^{op}}$ is fundamental. In this section, we will elaborate. The Yoneda Lemma first put forward by Nobuo Yoneda arises everywhere and has the reputation of being one of the most fundamental theorems in category theory.

Introduction

- This section describes two ideas.
- First, the Yoneda Embedding Theorem shows the category $\mathbb{A}$ embeds or “sits nicely” inside $\mathbb{Set}^{\mathbb{A}^{op}}$.
- To do this, we will identify the objects of $\mathbb{A}$ with certain objects of $\mathbb{Set}^{\mathbb{A}^{op}}$.
- Secondly, the Yoneda Lemma shows that every object in $\mathbb{Set}^{\mathbb{A}^{op}}$ is determined by the way it interacts with the objects from $\mathbb{A}$.
- The implications of these two ideas will be elaborated at the end of the section.

The Yoneda Embedding Theorem

Let us begin with $\mathbb{Set}^{\mathbb{A}^{op}}$. The objects of this category are functors $F\colon \mathbb{A}^{op} \longrightarrow \mathbb{Set}$ (we will see that such a functor is called a **presheaf**) and the morphisms are natural transformations of such functors. We met special functors of this type $Hom_{\mathbb{A}}(\ \ , a)\colon \mathbb{A}^{op} \longrightarrow \mathbb{Set}$ where a is an object in $\mathbb{A}$. Any functor that is isomorphic to such a functor is called a **representable functor**. In a sense, these functors are "represented" by the objects in $\mathbb{A}$.

The Yoneda Embedding Theorem

There is a functor $y_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{S}\mathrm{et}^{\mathbb{A}^{op}}$, called the **Yoneda embedding functor**, that takes every object $a \in \mathbb{A}$ to the functor it represents, i.e., $y_{\mathbb{A}}(a) = Hom_{\mathbb{A}}(\ \ , a)$. The functor $y_{\mathbb{A}}$ is defined on morphisms as follows: for every $f : a \longrightarrow a'$ in $\mathbb{A}$ there is a natural transformation

$$y_{\mathbb{A}}(f) : Hom_{\mathbb{A}}(\ \ , a) \Longrightarrow Hom_{\mathbb{A}}(\ \ , a').$$

For object b in $\mathbb{A}$, the b component is

$$y_{\mathbb{A}}(f)_b : Hom_{\mathbb{A}}(b, a) \longrightarrow Hom_{\mathbb{A}}(b, a')$$

and is defined as

$$k : b \longrightarrow a \qquad \longmapsto \qquad f \circ k : b \longrightarrow a \longrightarrow a'.$$

The Yoneda Embedding Theorem

One can see that $y_{\mathbb{A}}(f)$ is natural by showing that the following square is commutative for any $g\colon b' \longrightarrow b$

$$\begin{array}{ccc} Hom_{\mathbb{A}}(b,a) & \xrightarrow{\;y_{\mathbb{A}}(f)_b\;} & Hom_{\mathbb{A}}(b,a') \\ {\scriptstyle Hom_{\mathbb{A}}(g,a)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathbb{A}}(g,a')} \\ Hom_{\mathbb{A}}(b',a) & \xrightarrow[\;y_{\mathbb{A}}(f)_{b'}\;]{} & Hom_{\mathbb{A}}(b',a'). \end{array}$$

For a given $h\colon b \longrightarrow a$ in the upper left-hand corner, both paths around the square go to $f \circ h \circ g\colon b' \longrightarrow a'$ in the lower right-hand corner.

The Yoneda Embedding Theorem

Exercise

Show that $y_{\mathbb{A}}$ respects composition. That is, show for $f\colon a \longrightarrow a'$ and $f'\colon a' \longrightarrow a''$ in $\mathbb{A}$, we have $y_{\mathbb{A}}(f' \circ f) = y_{\mathbb{A}}(f') \circ_V y_{\mathbb{A}}(f)$.

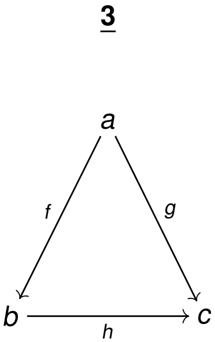
Exercise

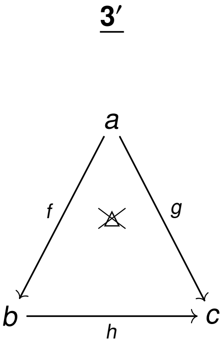
Show that $y_{\mathbb{A}}$ respects identity morphisms. That is, show $y_{\mathbb{A}}(id_a) = Id_{Hom(\ \ ,a)}$.

These two exercises show that $y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{Set}^{\mathbb{A}^{op}}$ is indeed a functor.

The Yoneda Embedding Theorem

Some examples of this functor are needed. Consider the following two categories.





These two categories are almost exactly alike except $\mathbf{3}'$ does not commute. We place the $\not\curlywedge$ to symbolize this failure of commutativity. Formally, this means that $g = h \circ f$ in $\mathbf{3}$ but $g \neq h \circ f$ in $\mathbf{3}'$. The next slide lists the representable functors for these categories.

$\mathbf{3}$	$\mathbf{3}'$
$Hom_{\mathbf{3}}(x,a) = \begin{cases} \{id_a\} & : \text{if } x = a \\ \emptyset & : \text{if } x = b \\ \emptyset & : \text{if } x = c \end{cases}$	$Hom_{\mathbf{3}'}(x,a) = \begin{cases} \{id_a\} & : \text{if } x = a \\ \emptyset & : \text{if } x = b \\ \emptyset & : \text{if } x = c \end{cases}$
$Hom_{\mathbf{3}}(x,b) = \begin{cases} \{f\} & : \text{if } x = a \\ \{id_b\} & : \text{if } x = b \\ \emptyset & : \text{if } x = c \end{cases}$	$Hom_{\mathbf{3}'}(x,b) = \begin{cases} \{f\} & : \text{if } x = a \\ \{id_b\} & : \text{if } x = b \\ \emptyset & : \text{if } x = c \end{cases}$
$Hom_{\mathbf{3}}(x,c) = \begin{cases} \{g = hf\} & : \text{if } x = a \\ \{h\} & : \text{if } x = b \\ \{id_c\} & : \text{if } x = c \end{cases}$	$Hom_{\mathbf{3}'}(x,c) = \begin{cases} \{g, hf\} & : \text{if } x = a \\ \{h\} & : \text{if } x = b \\ \{id_c\} & : \text{if } x = c \end{cases}$

The representable functors for **3** and **3′**.

The Yoneda Embedding Theorem

The morphisms of **3** and **3′** induce morphisms of representable functors. Let us examine the morphisms induced by $h\colon b \longrightarrow c$.

3

$$Hom_{\mathbf{3}}(\ ,b) \xrightarrow{y_{\mathbf{3}}(h)} Hom_{\mathbf{3}}(\ ,c)$$

$$f \longmapsto^{y_{\mathbf{3}}(h)_a} g = hf$$

$$id_b \longmapsto^{y_{\mathbf{3}}(h)_b} h$$

$$\{\ \} \longmapsto^{y_{\mathbf{3}}(h)_c} id_c$$

3′

$$Hom_{\mathbf{3'}}(\ ,b) \xrightarrow{y_{\mathbf{3'}}(h)} Hom_{\mathbf{3'}}(\ ,c)$$

$$f \longmapsto^{y_{\mathbf{3'}}(h)_a} hf$$

$$id_b \longmapsto^{y_{\mathbf{3'}}(h)_b} h$$

$$\{\ \} \longmapsto^{y_{\mathbf{3'}}(h)_c} id_c$$

The main point of these calculations is to show that all the information about the categories **3** and **3′** are in the representable functors and in the induced morphisms of the representable functors. This is the central idea of the Yoneda Embedding Theorem: a locally small category can be totally described as representable functors. Thus, a version of the category **3** "sits inside" the category $\mathbb{Set}^{\mathbf{3}^{op}}$, and a version of the category **3′** "sits inside" the category $\mathbb{Set}^{\mathbf{3}'^{op}}$. This idea is true for any small category.

The Yoneda Embedding Theorem

Theorem

The Yoneda Embedding Theorem *For any locally small category $\mathbb{A}$, the covariant functor $y_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{Set}^{\mathbb{A}^{op}}$ is injective on objects, full, and faithful.*

The Yoneda Embedding Theorem

Proof.

The functor is

- Injective on objects. Let $y_{\mathbb{A}}(a) = y_{\mathbb{A}}(a')$. This means that they are equal on every component including a. Hence $y_{\mathbb{A}}(a)_a = y_{\mathbb{A}}(a')_a$, which implies that $Hom(a, a) = Hom(a, a')$. The first set contains $id_a \colon a \longrightarrow a$. Since that morphism is in the second Hom set, a must equal a'.
- Full. Consider an arbitrary natural transformation $\beta \colon Hom(\ \ , a) \Longrightarrow Hom(\ \ , a')$. The naturality of β means that for any $k \colon b \longrightarrow a$, the following square commutes:

$$\begin{array}{ccc} Hom_{\mathbb{A}}(a, a) & \xrightarrow{\quad \beta_a \quad} & Hom_{\mathbb{A}}(a, a') \\ {\scriptstyle Hom(f,a)} \downarrow & & \downarrow {\scriptstyle Hom(f,a')} \\ Hom_{\mathbb{A}}(b, a) & \xrightarrow[\quad \beta_b \quad]{} & Hom_{\mathbb{A}}(b, a'). \end{array}$$

Continued.

In the upper left corner is id_a. The top horizontal map takes id_a to some map $f\colon a \longrightarrow a'$. The right vertical map takes f to $f \circ k$. The left vertical map takes id_a to $k\colon b \longrightarrow a$. In order for this square to commute β_b must take k to $f \circ k$. But this is exactly the definition of $y_{\mathbb{A}}(f\colon a \longrightarrow a')$. This means that $\beta = y_{\mathbb{A}}(f\colon a \longrightarrow a')$.

- Faithful. If $y_{\mathbb{A}}(f) = y_{\mathbb{A}}(f')$ then they are equal at the a component, i.e., $y_{\mathbb{A}}(f)_a = y_{\mathbb{A}}(f')_a$. This means that they have the same value at id_a. We have that $f = f \circ id_a = f' \circ id_a = f'$.

□

It is important to keep in mind that $y_{\mathbb{A}}$ is a covariant functor, but each of its images is a contravariant functor.

The Yoneda Embedding Theorem

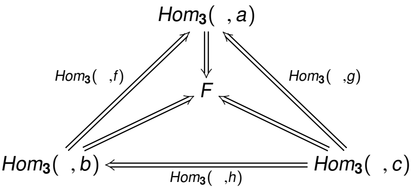
A theorem that shows that a structure can be represented as another structure is usually called a **representation theorem**. Perhaps the Yoneda Embedding Theorem should be called the **Yoneda Representation Theorem**.

Exercise

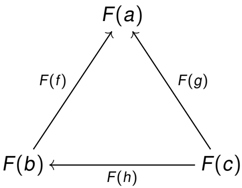
Consider the **contravariant Yoneda embedding functor** $y^{\mathbb{A}} \colon \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{et}^{\mathbb{A}}$ *that takes* a *to* $Hom_{\mathbb{A}}(a, \ )$ *and* $f \colon a \longrightarrow a'$ *to* $y^{\mathbb{A}}(f) = Hom_{\mathbb{A}}(a, f)$. *Show that* $y^{\mathbb{A}}(f)$ *is natural. Also show that* $y^{\mathbb{A}}$ *is injective on objects, full, and faithful. The contravariant Yoneda embedding says that* $\mathbb{A}^{op}$ *"sits nicely inside"* $\mathbb{S}\mathrm{et}^{\mathbb{A}}$. *Keep in mind that* $y^{\mathbb{A}}$ *is contravariant, but each of its images is a covariant functor.*

The main idea of the Yoneda lemma is not only that a small category embeds nicely inside its functor category, but other elements in its functor category are determined by their interaction with the embedded category. As an example, consider the category $\mathbf{3}$. We have the embedded representable functors and for every $F\colon \mathbf{3}^{op} \longrightarrow \mathbb{Set}$, there are natural tranformations from the representable functors to F. We can envision such maps as the left diagram:

$\underline{\mathbb{Set}^{\mathbf{3}^{op}}}$



$\underline{\mathbb{Set}}$



We will see that for a functor $F\colon \mathbb{A}^{op} \longrightarrow \mathbb{Set}$, the maps from the representable functors to F determine the values of F shown on the right. In other words, the representable functors determine all the functors.

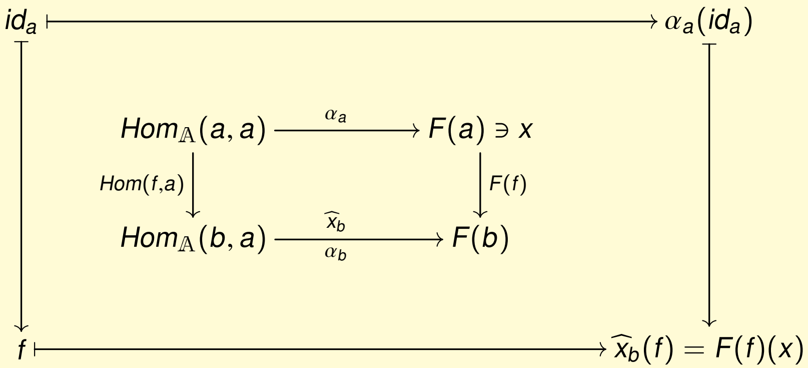
Theorem

The Yoneda Lemma. *For every locally small category $\mathbb{A}$, and for every $F\colon \mathbb{A}^{op} \longrightarrow \mathbb{Set}$, the set of natural transformations from $Hom_{\mathbb{A}}(\ \ , a)$ to F is naturally isomorphic with the set $F(a)$, i.e.,*

$$Hom_{\mathbb{Set}^{\mathbb{A}^{op}}}(Hom_{\mathbb{A}}(\ \ , a), F) \quad \cong \quad F(a).$$

Proof.

We describe an isomorphism $\Theta\colon Hom_{\mathbb{Set}^{\mathbb{A}^{op}}}(Hom_{\mathbb{A}}(\ \ ,a),F) \longrightarrow F(a)$. The following diagram will be helpful.



□

Continued.

For a natural transformation $\alpha\colon Hom_{\mathbb{A}}(\ \ ,a) \Longrightarrow F$, we define $\Theta(\alpha) = \alpha_a(id_a)$. In detail, $\alpha_a : Hom_{\mathbb{A}}(a,a) \longrightarrow F(a)$ and $id_a \in Hom_{\mathbb{A}}(a,a)$, so $\alpha_a(id_a) \in F(a)$. The inverse of Θ is $\Theta^{-1}\colon F(a) \longrightarrow Hom_{\mathbb{Set}^{\mathbb{A}^{op}}}(Hom_{\mathbb{A}}(\ \ ,a), F)$. For an element x in the set $F(a)$, we define $\Theta^{-1}(x) = \widehat{x}$ where $\widehat{x}\colon Hom_{\mathbb{A}}(\ \ ,a) \Longrightarrow F$. For $b \in \mathbb{A}$, the component $\widehat{x}_b\colon Hom_{\mathbb{A}}(b,a) \longrightarrow F(b)$ is defined for a map $f\colon b \longrightarrow a$ in $\mathbb{A}$ as $\widehat{x}_b(f) = F(f)(x)$. In detail, $F(f)\colon F(a) \longrightarrow F(b)$ and $x \in F(a)$, so $F(f)(x) \in F(b)$. □

Continued.

It remains to show that Θ and Θ^{-1} are inverses.

- $\Theta(\Theta^{-1}(x)) = \Theta(\widehat{x}) = \widehat{x}_a(id_a) = F(id_a)(x) = id_{F(a)}(x) = x$.
- $\Theta^{-1}(\Theta(\alpha)) = \Theta^{-1}(\alpha_a(id_a)) = \widehat{\alpha_a(id_a)}$. Let us see how this natural transformation is defined at component b. The morphism $\widehat{\alpha_a(id_a)}_b$ is defined for $f \colon b \longrightarrow a$ as $F(f)(\alpha_a(id_a))$. By the naturality of α, this is exactly α_b. Thus we have shown that $\Theta^{-1}(\Theta(\alpha)) = \alpha$.

$\square$

Notice that the fact that $y_{\mathbb{A}}$ is full and faithful is basically a consequence of the Yoneda lemma. To see this, set F of the Yoneda lemma to $Hom_{\mathbb{A}}(\ \ , a')$. This gives us

$$Hom_{\mathbb{Set}^{\mathbb{A}^{op}}}(Hom_{\mathbb{A}}(\ \ , a), Hom_{A}(\ \ , a')) \quad \cong \quad Hom_{\mathbb{A}}(a, a').$$

Example

What does the Yoneda embedding say about partial orders? Let P be a partial order. Then $Hom_P(p, p')$ is either a one-element set (if $p \leq p'$) or the empty set (if $p \not\leq p'$). The functor $Hom_p(\ \ , p)$ tells whether or not any element is less than or equal to p. Putting this all together, gives us the following obvious property of partial orders:

$$[\ \text{For all } q \in P,\ \text{ if } q \leq p \text{ then } q \leq p'] \qquad \text{if and only if} \qquad p \leq p'.$$

($\Longleftarrow$ is obvious and $\Longrightarrow$ is true by setting $q = p$.)

There is a dual to the Yoneda Lemma:

Theorem

The **contravariant Yoneda Lemma**. *For every category* $\mathbb{A}$, *and for every* $F\colon \mathbb{A} \longrightarrow \mathbb{S}\mathrm{et}$, *the set of natural transformations from* $Hom_{\mathbb{A}}(a, \ )$ *to* F *is naturally isomorphic with the set* $F(a)$, *i.e.,*

$$Hom_{\mathbb{S}\mathrm{et}^{\mathbb{A}}}(Hom_{\mathbb{A}}(a, \ ), F) \quad \cong \quad F(a).$$

The Importance of the Yoneda Lemma

Why are the Yoneda Embedding Theorem and the Yoneda Lemma so important? They formalize ideas that we met many times:

- Way back in Chapter 1, we saw that elements in a set, S, can be described by morphisms $\{*\} \longrightarrow S$.
- Similarly, to find triplets in S, one should look at morphisms $\{a, b, c\} \longrightarrow S$.
- We also saw that objects in a graph are determined by graph homomorphisms from trivial graphs.
- Certain types of paths in a graph are determined by certain types of graph homomorphisms to the graph.
- Vectors in a $\mathbf{K}$-vector space V are described with linear transformations $\mathbf{K} \longrightarrow V$.
- We will see later that paths in a topological space (or manifold) T are determined by maps $[0, 1] \longrightarrow T$.

The Importance of the Yoneda Lemma

In all these examples, properties of an object in a category are determined by the maps to that object. The maps probe the objects. Both the Yoneda Embedding Theorem and the Yoneda Lemma show the full power of category theory by showing that the properties of an object are totally determined by the maps to the object. In other words, one should study the morphisms of a category to understand the structure of the objects in the category. This is the core of category theory.

The Importance of the Yoneda Lemma

- There is another way to think about the Yoneda Lemma.
- It says that every object in $\mathbb{Set}^{\mathbb{A}^{op}}$ can be written in a universal way as a bunch of maps from representable functors.
- Another way to say this is that an arbitrary element of $\mathbb{Set}^{\mathbb{A}^{op}}$ can be written as a colimit of representable functors.

The Importance of the Yoneda Lemma

- As we saw, the category $\mathbb{Set}$ is (complete and) cocomplete. The category $\mathbb{Set}^{\mathbb{A}^{op}}$ inherits this cocompleteness.
- The Yoneda Lemma says that the best (smallest) category that cocompletes $\mathbb{A}$ is $\mathbb{Set}^{\mathbb{A}^{op}}$.
- Another way to say this is that if you freely add colimits to $\mathbb{A}$, you get a category equivalent to $\mathbb{Set}^{\mathbb{A}^{op}}$.
- This functor category is the cocompletion of $\mathbb{A}$. A consequence of this is that if $\mathbb{A}$ starts off cocomplete, then $\mathbb{A}$ is equivalent to $\mathbb{Set}^{\mathbb{A}^{op}}$.

Mini-course:

Basic Categorical Logic

Foreshadowing

- Chapter 4: Relationships Between Categories
  - Section 4.8: Mini-course: Basic Categorical Logic
    - Propositional Logic
    - Predicate Logic

Foreshadowing

- We describe how category theorists look at the structures of logic.
- The first part is concerned with propositional logic which deals with true or false statements about properties of particular objects.
- The second part is concerned with predicate logic which deals with general properties of many objects.

History

The vast majority of material in this mini-course on categorical logic was first formulated by F. William Lawvere (1937 – 2023)

Definition

Logic is about **propositions** *which are statements that are true or false.* **Propositional logic** *is concerned with statements about properties of single entities.*

Example

- *"Category theory is easy" is a true proposition.*
- *"Category theory is boring" is totally false.*
- *"George Washington was a king of France" is false.*
- *"2 + 2 = 4" is true.*

Reminder

Our central focus will be the category of propositions or $\mathbb{P}\mathrm{rop}$.

A Category Defined

The preorder category, $\mathbb{P}\mathrm{rop}$, has propositions as objects, and there is a single morphism from proposition P to proposition Q iff P implies (or entails) Q.

Example

- *“Joan studies category theory” $\longrightarrow$ “Joan will be able to learn a lot of science.”*
- *“Joan will be able to learn a lot of science” $\longrightarrow$“Joan will be happy.”*
- *Combining these two implications means that “Joan studies category theory” $\longrightarrow$ “Joan will be happy.”*

Foreshadowing

Logic is not only about propositions but about operations on propositions and how propositions relate to each other. We will see the following operations

- Conjunction “and” $\wedge$
- Implication “implies” $\Longrightarrow$
- Disjunction “or” $\vee$
- Negation “not” $\neg$
- Bi-implication “iff” $\Leftrightarrow$.

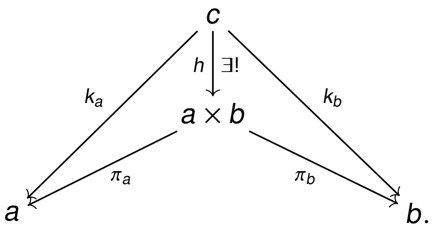
Conjunction

In $\mathbb{Prop}$, the **conjunction** or the **logical and** of the proposition is used as follows:

Example

- *"Jack will be able to learn a lot of physics" and the proposition*
- *"Jack will be able to learn a lot of mathematics" is the proposition*
- *"Jack will be able to learn a lot of physics"* $\wedge$ *"Jack will be able to learn a lot of mathematics."*
- *We can write this as"Jack will be able to learn a lot of physics* <u>*and*</u> *mathematics."*

Remember the definition of a product

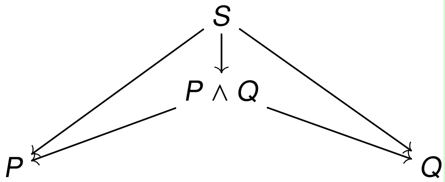


Definition

- *Let $\mathbb{A}$ be a category with objects a and b.*
- *A* **product** *of a and b is an object $a \times b$ with* **projection morphisms** π_a *and* π_b.
- *These satisfy the following universal property: for every object c and any two maps k_a and k_b,*
- *there exists a unique map $h\colon c \longrightarrow a \times b$ which makes both triangles commute.*

Conjunction

This $\wedge$ operation is the product in $\mathbb{Prop}$.



Definition (Product in $\mathbb{Prop}$.)

- *The product / conjunction of propositions P and Q is written as $P \wedge Q$.*
- *It is obvious that $P \wedge Q \longrightarrow P$ and $P \wedge Q \longrightarrow Q$. These are projections.*
- *To show that $P \wedge Q$ satisfies the universal property of being a product, realize that if $S \longrightarrow P$ and $S \longrightarrow Q$, then we can conclude $S \longrightarrow P \wedge Q$.*

Example

Projections maps:

- *"Jack will be able to learn a lot of physics and mathematics"* $\longrightarrow$ *"Jack will be able to learn a lot of physics" and*
- *"Jack will be able to learn a lot of physics and mathematics"* $\longrightarrow$ *"Jack will be able to learn a lot of mathematics."*

Example (Continued)

This proposition also satisfies the universal property of being a product. Consider any proposition that implies the two propositions.

- *"Jack is studying category theory"* $\longrightarrow$ *"Jack will be able to learn a lot of physics" and*
- *"Jack is studying category theory"* $\longrightarrow$ *"Jack will be able to learn a lot of mathematics."*

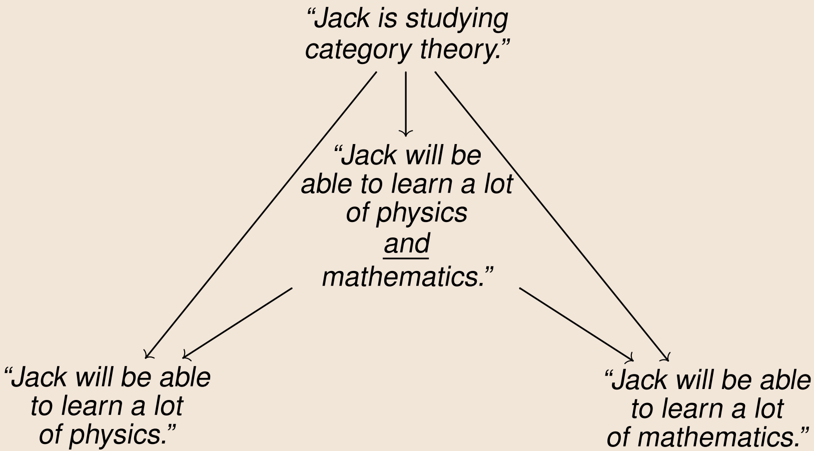
From this it is obvious that

- *"Jack is studying category theory"* $\longrightarrow$ *"Jack will be able to learn a lot of physics <u>and</u> mathematics."*

Conjunction

An example demonstrating the universal property of conjunction.

Example



Conjunction

- In terms of categories, the conjunction operations is a bifunctor
$$\wedge : \mathbb{P}\mathrm{rop} \times \mathbb{P}\mathrm{rop} \longrightarrow \mathbb{P}\mathrm{rop}.$$
- The functoriality of this operations means that if $P \longrightarrow P'$ and $Q \longrightarrow Q'$, then $(P \wedge Q) \longrightarrow (P' \wedge Q')$.
- We can describe the universal property of the product as
$$Hom_{\mathbb{P}\mathrm{rop}}(S, P \wedge Q) \cong Hom_{\mathbb{P}\mathrm{rop}}(S, P) \times Hom_{\mathbb{P}\mathrm{rop}}(S, Q).$$
The left side has a single arrow if both of the right Hom sets have a single element and neither of them is the empty set.

Conjunction

- Let us restate this using the product of Hom sets.

$$Hom_{\mathbb{Prop}}(S, P \wedge Q) \cong Hom_{\mathbb{Prop}^2}((S, S), (P, Q)).$$

- Using the diagonal functor $\Delta\colon \mathbb{Prop} \longrightarrow \mathbb{Prop}^2$, we get

$$Hom_{\mathbb{Prop}}(S, P \wedge Q) \cong Hom_{\mathbb{Prop}^2}(\Delta(S), (P, Q)).$$

- We have just proved that $\wedge\colon \mathbb{Prop}^2 \longrightarrow \mathbb{Prop}$ is right adjoint to $\Delta\colon \mathbb{Prop} \longrightarrow \mathbb{Prop}^2$, i.e.,

$$\mathbb{Prop} \underset{\wedge}{\overset{\Delta}{\rightleftarrows}}\ \perp\ \mathbb{Prop}^2.$$

Implication

Another operation on propositions is the **logical implication**. This takes two propositions and forms a conditional if-then statement out of them.

Example

- *Given the proposition "Jordan understands category theory" and the proposition "Jordan will see the world in a new light," we can form the proposition "If Jordan understands category theory, then he will see the world in a new light.'*
- *The proposition "It is raining" implies "There are clouds in the sky." This gives us the proposition "If it is raining, then there are clouds in the sky."*

Implication

Example

- *The proposition "Drive more than 60 miles per hour" might imply "Getting a speeding ticket" and so we can form the proposition "If you drive more than 60 miles per hour, then you will get a speeding ticket."*
- *The proposition "If the moon is made of green cheese, then the sky is blue." is an implication. Notice the first part really has nothing to do with the second part. The implication is true because the sky is blue.*

Implication

- In symbols, given proposition P and Q, we can form $P \Rightarrow Q$.
- This statement is true if P implies Q.
- It is important to keep in mind the distinction between $\Rightarrow$ and $\longrightarrow$.
- The arrow $\Rightarrow$ is an operation of two propositions and is a logical symbol.
- The arrow $\longrightarrow$ is a categorical symbol that describes when there is an implication of propositions in $\mathbb{P}\mathrm{rop}$.

Implication

- In terms of categories, the $\Rightarrow$ operation can be seen as a bifunctor
$$\Rightarrow : \mathbb{P}\mathrm{rop}^{op} \times \mathbb{P}\mathrm{rop} \longrightarrow \mathbb{P}\mathrm{rop}.$$
- It is covariant on the second input: If $Q \longrightarrow Q'$, then $(P \Rightarrow Q) \longrightarrow (P \Rightarrow Q')$.
- It is contravariant in the first input: If $P \longrightarrow P'$ then $(P' \Rightarrow Q) \longrightarrow (P \Rightarrow Q)$.

Conjunction and Implication

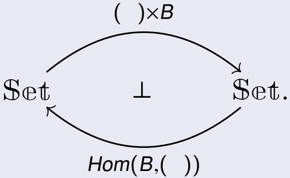
The $\wedge$ and $\Rightarrow$ operations are related.

Reminder (Similar functors on $\mathbb{S}\mathrm{et}$)

- *For every set B, there are two functors from $\mathbb{S}\mathrm{et}$ to $\mathbb{S}\mathrm{et}$: $L_B(A) = A \times B$ and $R_B(C) = Hom_{\mathbb{S}\mathrm{et}}(B, C)$.*
- *The functor L_B is left adjoint to R_B:*

$$Hom_{\mathbb{S}\mathrm{et}}(L_B(A), C) \cong Hom_{\mathbb{S}\mathrm{et}}(A, R_B(C)).$$

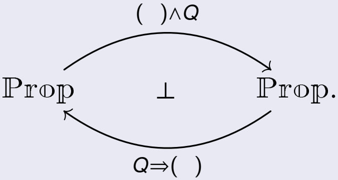
- $$Hom_{\mathbb{S}\mathrm{et}}(A \times B, C) \cong Hom_{\mathbb{S}\mathrm{et}}(A, Hom_{\mathbb{S}\mathrm{et}}(B, C)).$$

- 

- For every proposition Q, there is a a functor $(\ \ ) \wedge Q \colon \mathbb{P}\mathrm{rop} \longrightarrow \mathbb{P}\mathrm{rop}$ that is defined on proposition P as $P \wedge Q$.
- For every proposition Q, there is a functor $Q \Rightarrow (\ \ ) \colon \mathbb{P}\mathrm{rop} \longrightarrow \mathbb{P}\mathrm{rop}$ which is defined on proposition S as $Q \Rightarrow S$.

Theorem

For every proposition Q, the functor $(\ \ ) \wedge Q$ is left adjoint to $Q \Rightarrow (\ \ )$, i.e.,



Conjunction and Implication

- The adjunction means
$$Hom_{\mathbb{P}rop}(P \wedge Q, S) \cong Hom_{\mathbb{P}rop}(P, Q \Rightarrow S).$$
- Or
$$(P \wedge Q) \longrightarrow S \qquad \text{if and only if} \qquad P \longrightarrow (Q \Rightarrow S).$$
- In English, if $P \wedge Q$ implies S, then just P itself implies that if Q is also true, then S is true. The other direction is similar.
- It pays to examine the unit and counit of this adjunction.
- The unit is $P \longrightarrow (Q \Rightarrow (P \wedge Q))$. This says that if P is true, then Q not only implies Q but also implies P.
- The counit is a little more famous. It says
$$(Q \Rightarrow P) \wedge Q \longrightarrow P.$$
In English, the counit expresses the fact that if Q implies P, and Q is true, then P is also true. This rule is called **modus ponens** which means the "way of pushing." We are pushing the implication forward.

Disjunction

- Another operation on propositions is the **disjunction** operation $\vee$, or the **logical or**operation.
- In symbols, given propositions P and Q, we can form proposition $P \vee Q$.

Example

Given proposition "Joan is good at category theory" and the proposition "2+2=4," we can form the proposition "Joan is good at category theory or 2+2=4."

Disjunction

- In terms of categories, the disjunction operation is a bifunctor

$$\vee : \mathbb{P}\mathrm{rop} \times \mathbb{P}\mathrm{rop} \longrightarrow \mathbb{P}\mathrm{rop}.$$

- The functoriality means that if $P \longrightarrow P'$ and $Q \longrightarrow Q'$, then $(P \vee Q) \longrightarrow (P' \vee Q')$.

Disjunction

The disjunction operation $\vee$ is the coproduct in the category $\mathbb{P}\mathrm{rop}$.

Example

Inclusion maps:

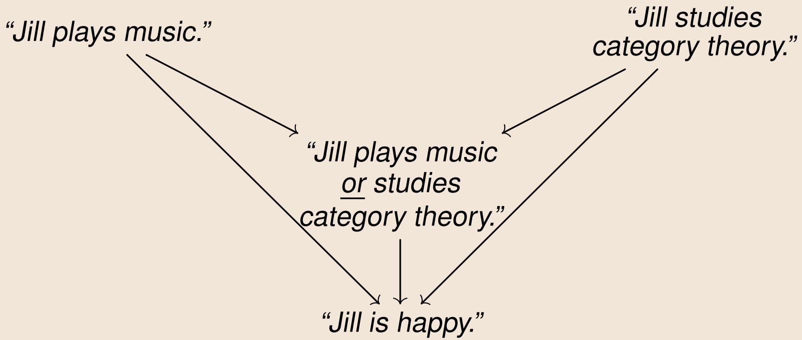
- *Consider the propositions "Jill plays music" and "Jill studies category theory."*
- *The disjunction of these propositions is "Jill plays music or studies category theory."*
- *There are obvious implications (inclusions) "Jill plays music" $\longrightarrow$ "Jill plays music or studies category theory" and*
- *"Jill studies category theory." $\longrightarrow$ "Jill plays music or studies category theory."*

Example (Continued)

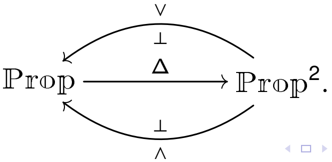
- *In order to show that the disjunction is the coproduct, we must show that it satisfies the universal property.*
- *Consider the proposition "Jill is happy."*
- *There are obvious implications "Jill plays music" $\longrightarrow$ "Jill is happy" and*
- *"Jill studies category theory" $\longrightarrow$ "Jill is happy."*
- *From this we can see that "Jill plays music or studies category theory" $\longrightarrow$ "Jill is happy."*

An example showing the universal property of $\vee$ as a coproduct.

Example



- We can describe the universal property of the coproduct as
$$Hom_{\mathbb{Prop}}(P \vee Q, S) \cong Hom_{\mathbb{Prop}}(P, S) \times Hom_{\mathbb{Prop}}(Q, S).$$
- The left side has a single arrow if both of the right Hom sets have a single element and neither of them is the empty set.
- Restate this using the product of Hom sets:
$$Hom_{\mathbb{Prop}}(P \vee Q, S) \cong Hom_{\mathbb{Prop}^2}((P, Q), (S, S)).$$
- Restate with the diagonal functor $\Delta\colon \mathbb{Prop} \longrightarrow \mathbb{Prop}^2$:
$$Hom_{\mathbb{Prop}}(P \vee Q, S) \cong Hom_{\mathbb{Prop}^2}((P, Q), \Delta(S)).$$
- We have just proved that $\vee\colon \mathbb{Prop}^2 \longrightarrow \mathbb{Prop}$ is left adjoint to $\Delta\colon \mathbb{Prop} \longrightarrow \mathbb{Prop}^2$.



Negation

- Another operation is the **negation** operation or the **logical not** operation.
- For the proposition "Jack studied category theory" the negation is "Jack did not study category theory."
- In symbols, for proposition P, we write the negation as $\neg P$.
- We can also define negation as the following $\neg P = P \Rightarrow F$ where F is the proposition that is always false.

Negation

- With this definition, we view the negation operation as a contravariant functor

$$\neg : \mathbb{P}\text{rop}^{op} \longrightarrow \mathbb{P}\text{rop}.$$

- The functoriality of negation means that if $P \longrightarrow P'$, then $(\neg P') \longrightarrow (\neg P)$. This says that if an implication is true, so is its contrapositive.
- One can "internalize" this result, where it becomes

$$((P \Rightarrow P') \wedge (\neg P')) \longrightarrow (\neg P).$$

- This rule is also one of the important rules in logic and is known as **modus tollens** which is Latin for "the way of pulling." One is "pulling" the negation of the second proposition back to the first proposition.

Bi-implication

- The final logical operation that we introduce is the **bi-implication** or **logical if and only if**.
- Two propositions bi-imply each other if they each implies the other.
- We use the symbol $\Leftrightarrow$ for this operation.
- In symbols $P \Leftrightarrow Q$ is interpreted as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Example

"Today is Tuesday" $\Leftrightarrow$ *"Tomorrow is Wednesday."*

Definition

Consider composition of the negation operation with itself:

$$\mathbb{P}\mathrm{rop}^{op} \underset{\neg}{\overset{\neg}{\rightleftarrows}} \mathbb{P}\mathrm{rop}.$$

- *If these two functors form an adjunction, the logical systems is called* **intuitionistic**.
  - *The unit of the adjunction says that for every P in $\mathbb{P}\mathrm{rop}$, we have $P \longrightarrow \neg\neg P$.*
  - *The counit says that for all P, we have $\neg\neg P \longrightarrow P$ in $\mathbb{P}\mathrm{rop}^{op}$ which is equivalent to the unit in $\mathbb{P}\mathrm{rop}$.*
- *If these two functors form an equivalence, the logical system is called* **Boolean**.
  - *In a Boolean system, for a proposition P, we have $\neg\neg P \Leftrightarrow P$.*

DeMorgan's laws

Theorem

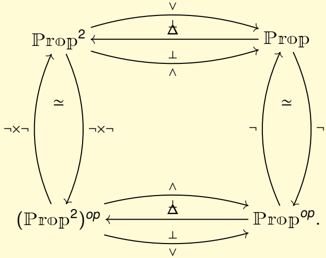
In a Boolean system, the following **DeMorgan's laws** *hold:*

$$(\neg P \vee \neg Q) \Leftrightarrow \neg(P \wedge Q) \qquad \text{and} \qquad (\neg P \wedge \neg Q) \Leftrightarrow \neg(P \vee Q).$$

DeMorgan's laws

Proof.

The proof all falls out of the adjunctions. We will go through this proof carefully as the rest of the proofs in this section are very similar.



- The vertical functors are equivalences and hence right and left adjoints to each other.
- The functors on bottom invert their adjointness because they are in the opposite category.

DeMorgan's laws

Continued.

The two paths from the lower-right corner to the upper-left corner are equal as can be seen here:

$$\begin{array}{ccc}
(\neg P, \neg P) & \longleftarrow\!\shortmid & \neg P \\
\uparrow & & \uparrow \\
(P, P) & \longleftarrow\!\shortmid & P.
\end{array}$$

They both take a proposition P in $\mathbb{Prop}^{op}$ to $(\neg P, \neg P)$ in $\mathbb{Prop}^2$. □

DeMorgan's laws

Continued.

Since any two right adjoints to this map are isomorphic, the two right adjoint paths from the upper-left corner to the lower-right corner are isomorphic. In terms of elements, this is

$$\begin{array}{ccc} P, Q & \longmapsto & P \wedge Q \\ \Big\downarrow & & \Big\downarrow \\ \neg P, \neg Q & \longmapsto & (\neg P \vee \neg Q) \cong \neg(P \wedge Q). \end{array}$$

This entails the first DeMorgan's law. □

Continued.

The left adjoints are also unique up to isomorphism and hence the isomorphism of the two left adjoint maps give the second DeMorgan's law. In detail this is

$$\begin{array}{ccc} P, Q & \longmapsto & P \vee Q \\ \Big\downarrow & & \Big\downarrow \\ \neg P, \neg Q & \longmapsto & (\neg P \wedge \neg Q) \cong \neg(P \vee Q). \end{array}$$

□

Lindenbaum category

While $\mathbb{Prop}$ is a preorder, the partial order associated with it is of importance also.

A Category Defined

The skeletal category of $\mathbb{Prop}$ *is a partial order, which we call the* **Lindenbaum category** *and denoted* $\mathbb{Lind}$. *The objects are equivalence classes of propositions. Two propositions* P *and* Q *are equivalent if* $P \Leftrightarrow Q$.

Keep in mind that $P \Leftrightarrow Q$ in $\mathbb{Prop}$ if and only if $P = Q$ in $\mathbb{Lind}$.

Predicate Logic

Logic gets much more interesting when one moves beyond dealing with the properties of particular entities and starts dealing with properties in general.

Definition

- *For every property, there is a* **predicate** *that tells which entities have this property.*
- *For every predicate there is a set of possible inputs to the predicate called the* **domain of discourse** *for the predicate.*

We begin with some fun examples.

Example

- *There is a predicate $H(x)$ that tells if someone is happy or not. So $H(Bill)$ is true if Bill is happy and false if Bill is not. The domain of discourse for H is the set of all people.*
- *Another example of a predicate is $KCT(x)$ which is true if x knows category theory and false if x does not know category theory. The domain of discourse for KCT is also the set of all people.*
- *There are also predicates with more than one input. For example $M(x, y)$ is the predicate that is true when x is the mother of y. The domain of discourse for M is the set of pairs of people.*

Here are some more mathematical examples:

Example

- $E(n)$ *is true when* n *is an even natural number. So* $E(1032)$ *is true but* $E(777)$ *is false. The domain of discourse for* E *is the set of natural numbers.*
- $P(n)$ *is true when* n *is a prime number. So* $P(7)$ *is true but* $P(57)$ *is false. The domain of discourse for* E *is the set of natural numbers.*
- $G(m, n)$ *is true when* m *is greater than* n*. The domain of discourse for* G *is pairs of natural numbers.*
- *The predicate* $A(x, n)$ *is true if person* x *is* n *years old. The domain of discourse for this predicate is the set of pairs of people and natural numbers.*

- One can think of a predicate as a function from the domain of discourse to the set {True, False}.
- A predicate describes a subset of the domain of discourse where the predicate is true.
- The subset that contains those elements of the domain of discourse that have the property.

Example

- *The predicate $H(x)$ describes the subset of people who are happy.*
- *The predicate $P(n)$ describes the subset of natural numbers that are prime.*

Predicate Logic

The same logical operations ($\wedge$, $\vee$, $\Rightarrow$, $\neg$, and $\Leftrightarrow$) that work for propositions work for predicates also. The operations are interpreted in the same way.

Example

- $KCT(x) \Rightarrow H(x)$ *says that if someone knows category theory, then they are happy.*
- $(M(x,y) \wedge KCT(y)) \Rightarrow H(x)$ *says that the mother of a child who knows category theory is happy.*
- $(A(x,m) \wedge A(y,n) \wedge M(x,y)) \Rightarrow G(m,n)$ *says that a mother's age is greater than their child's age.*
- $E(n) \Rightarrow \neg E(n+1)$ *says that if a number is even then its successor is not even.*
- $(G(n,2) \wedge P(n)) \Rightarrow \neg E(n)$ *says that every prime more than 2 is not even.*

Quantifiers

In addition to such logical operations, predicates have **quantifiers**. These are devices that tell the quantity "how much" or the quantity of elements of the domain of discourse. There are two main quantifiers:

- A **universal quantifier** written $\forall$. The predicate $\forall y Q(x, y)$ is true if and only if $Q(x, y)$ is true <u>for all</u> possible y in the domain of discourse.
- An **existential quantifier** written $\exists$. The predicate $\exists y Q(x, y)$ is true if and only if $Q(x, y)$ is true <u>for some</u> possible y in the domain of discourse.

Example

- $\forall m \exists n G(n, m)$. *This says that for all m, there is an n such that n is greater than m. In other words, the set of natural numbers is infinite.*
- $\exists n (E(n) \wedge P(n))$. *This says that there is a number that is even and is prime. This is true because 2 is such a number that is both even and prime.*
- $\forall n (P(n) \Rightarrow (\exists m G(m, n) \wedge P(m)))$. *This says that for any number, if it is a prime number, then there exists a larger prime number. In other words, the collection of prime numbers is infinite. This statement is true.*

Quantifiers

Example

- $\forall n((P(n) \wedge P(n+2)) \Rightarrow (\exists m G(m,n) \wedge (P(m) \wedge P(m+2))))$.
- *To understand this, one must understand the concept of twin primes.*
- *Two primes are twin if they are separated by one intermediate number.*
- *For example: 3 and 5; 11 and 13; 599 and 601.*
- *This statement says that if n and $n+2$ are twin primes, then there is a larger m such that m and $m+2$ are twin primes.*
- *This means there are an infinite amount of twin primes.*

It is not known if this statement is true or false.

Quantifiers

- Quantifiers work on predicates with many variables.
- Consider a set of variables $\{x_1, x_2, \ldots, x_n\}$. We can write this set as $\bar{x}$.
- A predicate with all these variables can be written as $P(\bar{x})$.
- A predicate $S(\bar{x}, y)$ has $n+1$ variables.
- Examples
  - $\forall y S(\bar{x}, y)$
  - $\exists y S(\bar{x}, y)$.
  - The predicate $\exists y \forall x \exists b B(a, b, c, d, x, y, z)$ uses seven variables. The variables y, x, and b are **bound** by quantifiers. The rest of the variables are not bound and are called **free**.

Let us look at predicate logic from the categorical point of view. Collections of predicates form preorder categories.

A Category Defined

- *Let* $\bar{x} = \{x_1, x_2, \ldots, x_n\}$ *be a set of variables.*
- *Then the collection of all predicates that have any of these variables as free variables forms a category called* $\mathbb{P}\mathrm{red}(\bar{x})$.
- *There is a morphism* $R(\bar{x}) \longrightarrow S(\bar{x})$ *whenever* R *is true (for its variables) implies* S *is true (for its variables).*

- The category $\mathbb{P}\mathrm{rop}$ of propositional statements is the category of $\mathbb{P}\mathrm{red}(\emptyset)$.
- If $\bar{x} \subseteq \bar{y}$ are two sets of variables, then a predicate $P(\bar{x})$ can also be seen as a predicate $P(\bar{y})$. This means that there is an inclusion of categories $\mathbb{P}\mathrm{red}(\bar{x}) \hookrightarrow \mathbb{P}\mathrm{red}(\bar{y})$.

Predicate Logic

The same way logical operations on propositions can be viewed as functors on $\mathbb{Prop}$, so too, logical operations on predicts can be seen as functors on $\mathbb{Pred}(\bar{x})$. These functors satisfy the same universal properties as they did with $\mathbb{Prop}$. This means that for $\bar{x}$, we have the following functors and adjunctions:

$$\mathbb{Pred}(\bar{x}) \underset{Q(\bar{x}) \Rightarrow (\ \ )}{\overset{(\ \ )\wedge Q(\bar{x})}{\rightleftarrows}} \mathbb{Pred}(\bar{x}). \qquad (\ \ )\wedge Q(\bar{x}) \dashv Q(\bar{x}) \Rightarrow (\ \ )$$

$$\vee \dashv \Delta \dashv \wedge, \qquad \Delta : \mathbb{Pred}(\bar{x}) \longrightarrow \mathbb{Pred}(\bar{x})^2, \quad \vee, \wedge : \mathbb{Pred}(\bar{x})^2 \longrightarrow \mathbb{Pred}(\bar{x}).$$

Predicate Logic

$$\mathbb{Pred}(\bar{x})^{op} \underset{\neg}{\overset{\neg}{\rightleftarrows}} \mathbb{Pred}(\bar{x}).$$

This diagram might be an adjunction or an equivalence depending on whether the logical system is intuitionistic or Boolean, respectively.

Quantifiers as functors

- Within predicate logic, the quantifiers also become functors with universal properties between the predicate categories.
- If $Q(\bar{x}, y)$ is a predicate in $\mathbb{Pred}(\bar{x}, y)$, then $\forall y Q(\bar{x}, y)$ is a predicate in $\mathbb{Pred}(\bar{x})$.
- The same is true for the predicate $\exists y Q(\bar{x}, y)$.
- These two mappings are functors

$$\forall y \colon \mathbb{Pred}(\bar{x}, y) \longrightarrow \mathbb{Pred}(\bar{x})$$

and

$$\exists y \colon \mathbb{Pred}(\bar{x}, y) \longrightarrow \mathbb{Pred}(\bar{x}).$$

- Functoriality comes from the fact that if $Q(\bar{x}, y) \longrightarrow R(\bar{x}, y)$ in $\mathbb{Pred}(\bar{x}, y)$, then

$$\forall y Q(\bar{x}, y) \longrightarrow \forall y R(\bar{x}, y)$$

and

$$\exists y Q(\bar{x}, y) \longrightarrow \exists y R(\bar{x}, y)$$

in $\mathbb{Pred}(\bar{x})$.

Quantifiers as adjoints

The universal and existential quantifiers are adjoints. However it takes a little effort to see this. First some reminders.

Reminder

- *Let S and T be sets. Then the powersets $\mathcal{P}(S)$ and $\mathcal{P}(T)$ form partial order categories.*
- *For any set function $f\colon S \longrightarrow T$, there is an induced functor $f^{-1}\colon \mathcal{P}(T) \longrightarrow \mathcal{P}(S)$ called the preimage or inverse image functorand is defined on a subset $Y \subseteq T$ as*

$$f^{-1}(Y) = \{x \quad : \quad f(x) = y \text{ for some } y \in Y\}.$$

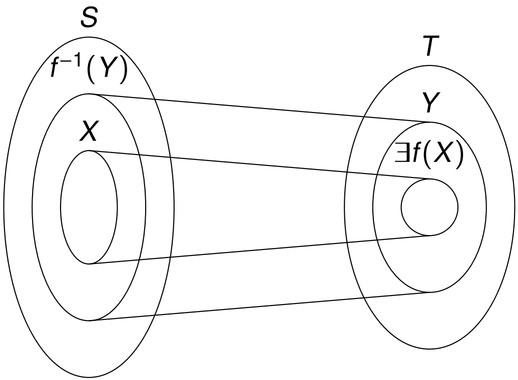
- *For any set function $f\colon S \longrightarrow T$, there is also an induced functor $\exists f\colon \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$ called the direct image functor ot the image functor and is defined on a subset $X \subseteq S$ as*

$$\exists f(X) = \{f(x) : x \in X\} = \{y : \text{there is an } x \in X \text{ and } f(x) = y\}.$$

Quantifiers as adjoints

- The direct image functor is left adjoint to the preimage functor.
- In other words, for $X \subseteq S$ and $Y \subseteq T$ there is the following isomorphism.

$$X \subseteq f^{-1}(Y) \text{ if and only if } \exists f(X) \subseteq Y.$$



Quantifiers as adjoints

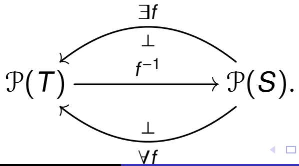
- For any set function $f\colon S \longrightarrow T$, there is yet another induced functor $\forall f\colon \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$, which we call the coimage functor and is defined for $X \subseteq S$ as

$$\forall f(X) = \{y \;:\; \text{for all } s \in S, \text{ if } f(s) = y, \text{ then } s \in X\}.$$

- In English, $\forall f(X)$ consists of those elements in T where all the preimages are only in X and nowhere else.
- The preimage functor f^{-1} is left adjoint to the coimage functor $\forall f$.
- This means that for $X \subseteq S$ and $Y \subseteq T$,

$$f^{-1}(Y) \subseteq X \text{ if and only if } Y \subseteq \forall f(X).$$

- These two adjunctions can be summarized with



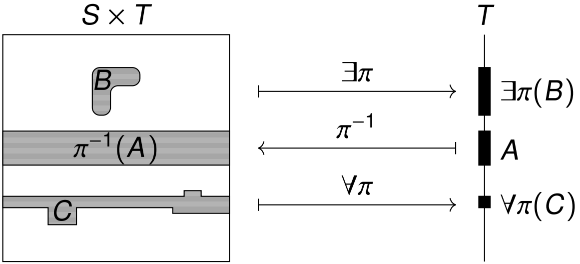
Quantifiers as adjoints

- Our goal is still to show that quantifiers are adjoints.
- To reach our goal, take two sets S and T and consider the projection function $\pi\colon S \times T \longrightarrow T$.
- π induces three functors: π^{-1}, $\exists\pi$, and $\forall\pi$.
- The functions are defined for $X \subseteq S \times T$ and $Y \subseteq T$ as
  - $\pi^{-1}(Y) = \{(s,t) \in S \times T \mid t \in Y\} \subseteq S \times T$
  - $\exists\pi(X) = \{t \in T \mid \text{ there exists a } (s,t) \in X \text{ and } \pi(s,t) = t\} \subseteq T$
  - $\forall\pi(X) = \{t \in T \mid \text{ for all } (s,t) \in X \text{ we have } \pi(s,t) = t\} \subseteq T$

Quantifiers as adjoints

How does the preimage, direct image, and coimage functors work for a projection function?

- The $\exists\pi$ projects B.
- The π^{-1} takes A to the whole strip.
- The $\forall\pi$ takes C to the part which has the entire strip highlighted.



Quantifiers as adjoints

These functors are adjoint as follows

$$\mathcal{P}(T) \underset{\underset{\forall\pi}{\longleftarrow}}{\overset{\overset{\exists\pi}{\longleftarrow}}{\xrightarrow{\pi^{-1}}}} \mathcal{P}(S \times T).$$

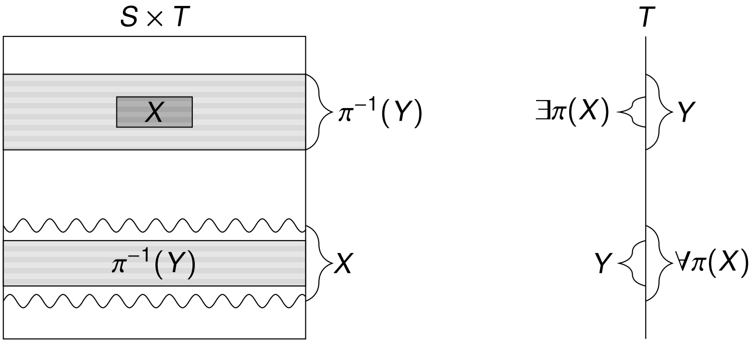
($\exists\pi \dashv \pi^{-1} \dashv \forall\pi$)

The two adjunctions can are written as the following statements:

$$X \subseteq \pi^{-1}(Y) \qquad \text{if and only if} \qquad \exists\pi(X) \subseteq Y$$

$$\pi^{-1}(Y) \subseteq X \qquad \text{if and only if} \qquad Y \subseteq \forall\pi(X).$$

The two adjunctions can be seen here:



At last, we come to show that quantifiers are adjoints.

- With the above adjunctions in mind, we can talk about functors between the categories of predicates.
- For every two variables x and y, there is an inclusion functor $\Delta_y \colon \mathbb{Pred}(x) \hookrightarrow \mathbb{Pred}(x, y)$.
- It takes a predicate $Q(x)$ to the predicate $Q(x, y)$ where the y variable is not used.
- We denote this inclusion functor Δ_y because it is like the diagonal functor in the sense that if $Q(x)$ is true, then $Q(x, y)$ is true *for all* possible values of y.
- This is reminiscent of the diagonal functor $\Delta \colon \mathbb{B} \longrightarrow \mathbb{B}^{\mathbb{A}}$, where $\Delta(b)$ is a functor whose output is b *for all* possible values of a in $\mathbb{A}$.

Quantifiers as adjoints

- The predicate $Q(x, y)$ describes a subset of pairs of elements of the domain of discourse.
- The predicate $Q(x)$ describes a subset of elements of the projection of the domain of discourse.
- The Δ_y functor is exactly the same as the π^{-1} of the previous discussion.
- As such, the Δ_y also has two adjoints.

Quantifiers as adjoints

- Let us boost this up from a single variable x to a set of variables $\bar{x}$.
- Given a predicate $Q(\bar{x})$ in $\mathbb{Pred}(\bar{x})$ and a predicate $R(\bar{x}, y)$ in $\mathbb{Pred}(\bar{x}, y)$, we have the following adjunctions:

$$R(\bar{x}, y)) \longrightarrow \Delta_y(Q(\bar{x})) \quad \text{if and only if} \quad \exists y R(\bar{x}, y)) \longrightarrow Q(\bar{x})$$

$$\Delta_y(Q(\bar{x})) \longrightarrow R(\bar{x}, y)) \quad \text{if and only if} \quad Q(\bar{x}) \longrightarrow \forall y R(\bar{x}, y)).$$

Quantifiers as adjoints

- Hence there are the following two adjunctions.

$$\mathbb{P}\mathrm{red}(\bar{x}) \xrightarrow{\;\Delta_y\;} \mathbb{P}\mathrm{red}(\bar{x}, y).$$
$$\exists_y : \mathbb{P}\mathrm{red}(\bar{x}, y) \to \mathbb{P}\mathrm{red}(\bar{x}) \quad \bot \qquad \forall_y : \mathbb{P}\mathrm{red}(\bar{x}, y) \to \mathbb{P}\mathrm{red}(\bar{x}) \quad \bot$$

- This diagram is similar to

$$\mathbb{P}\mathrm{rop} \xrightarrow{\;\Delta\;} \mathbb{P}\mathrm{rop}^2.$$
$$\vee : \mathbb{P}\mathrm{rop}^2 \to \mathbb{P}\mathrm{rop} \quad \bot \qquad \wedge : \mathbb{P}\mathrm{rop}^2 \to \mathbb{P}\mathrm{rop} \quad \bot$$

.

- This makes sense because the universal quantifier is a generalization of the $\wedge$, and the existential quantifier is a generalization of $\vee$.

Properties of quantifiers

Theorem

In a Boolean logical system, the following **generalized DeMorgan's laws** *hold:*

$$\neg\forall_y P(\bar{x}, y) \Longleftrightarrow \exists_y \neg P(\bar{x}, y)$$

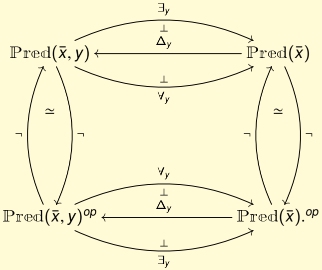
and

$$\neg\exists_y P(\bar{x}, y) \Longleftrightarrow \forall_y \neg P(\bar{x}, y).$$

Properties of quantifiers

Proof.

Consider the following square with a Boolean system of logic



The proof follows along the same line as the proof of regular DeMorgan's laws. □

Properties of quantifiers

Theorem

Quantifiers respect the appropriate operation:

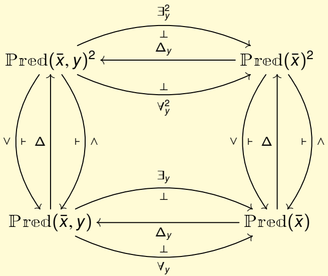
$$\exists x P(x) \vee \exists x Q(x) \iff \exists x (P(x) \vee Q(x))$$

and

$$\forall x P(x) \wedge \forall x Q(x) \iff \forall x (P(x) \wedge Q(x))$$

Proof.

Consider the following diagram:



The two maps from the lower right to the upper left amount to $Q(\bar{x}) \mapsto (\Delta_y Q(\bar{x}), \Delta_y Q(\bar{x}))$. This means that the two compositions of two right adjoints (and the two left) are isomorphic. □

Properties of quantifiers

Theorem

Quantifiers respect quantifiers

$$\exists y \exists z P(\bar{x}, y, z) \qquad \Longleftrightarrow \qquad \exists z \exists y P(\bar{x}, y, z)$$

and

$$\forall y \forall z P(\bar{x}, y, z) \qquad \Longleftrightarrow \qquad \forall z \forall y P(\bar{x}, y, z).$$

The proof is left as a (solved) Exercise in the book.

Logic and sets

Remark

It should be noticed that logic and sets are intimately related.

- *The intersection of two sets is the set of elements that are in one set and in the other set. In symbols, for sets S and T,*

$$S \cap T = \{x \quad : \quad x \in S \wedge x \in T\}.$$

- *The union of two sets is the set of elements that are in one set or the other set. In symbols, for sets S and T,*

$$S \cup T = \{x \quad : \quad x \in S \vee x \in T\}.$$

- *The complement of a set S is the set of elements that are not in S. In symbols, for set S,*

$$S^c = \{x \quad : \quad \neg(x \in S)\}.$$

Summary

It pays to meditate on what was accomplished here. What was gained by presenting logic from a categorical point of view?

Remark

- *Our goal was not simply to show that logic can be done in the language of category theory. Rather, our goal was to demonstrate several important ideas:*
- *We showed that logic is united with the other fields that we met in this course. The unity comes from the fact that logic employs the same tools of products, coproducts, functoriality, equivalences, adjunction, etc., that are used in other areas.*

Remark (Continued)

- *We showed that the various logical operations do not stand alone. Category theory shows that the operations are intimately connected to each other and can be defined in terms of each other with universal properties.*
- *We showed that many of the truths of propositional and predicate logic are simple consequences of the universal properties of the operations. A logical statement is not true because it seems true. Rather the statement has to be true because of the way operations are defined in terms of other operations.*

Categorical logic is a wonderful modern contribution to the ancient field of logic.